

The Lamb shift in muonic hydrogen and the proton radius from effective field theories

Based on:

Pineda, hep-ph/0210210, hep-ph/0308193, hep-ph/0412142

Nevado&Pineda, 0712.1294

Peset&Pineda, 1403.3408, 1406.4524, 1508.01948

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Muonic hydrogen

Scales:

$$m_p \sim \Lambda_\chi$$

$$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p + m_\mu}$$

$$m_r \alpha \sim m_e$$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r \alpha}{m_r} \sim \frac{m_r \alpha^2}{m_r \alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision: $m_r \alpha^5$ (Heavy Quarkonium)

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Tool:

Effective Field Theories \equiv Factorization

Why?: There is a hierarchy of different scales (hard, soft and ultrasoft).

$$m \gg mv \gg mv^2, \quad (\Lambda_{QCD})$$

EFTs are especially useful in these situations.

- 1) Perturbative calculations much easier and systematic.
- 2) Nonperturbative information is parameterized in a model independent way.
- 3) Power counting.
- 4) Connection between non-relativistic (NR) Quantum Mechanics and Quantum Field Theories.

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Theoretical setup (muonic hydrogen)

We use an effective field theory, **Potential Non-Relativistic QED**, which describes the muonic hydrogen dynamics and profits from the hierarchy

$$m_\mu \gg m_\mu \alpha \gg m_\mu \alpha^2$$

$$\left. \begin{aligned} & \left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r} \right) \psi(\mathbf{r}) = 0 \\ & + \text{corrections to the potential} \\ & + \text{interaction with ultrasoft photons} \end{aligned} \right\} \text{potential NRQED} \quad E \sim mv^2$$

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

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Summary (muonic hydrogen)

1) Matching HBET to NRQED. Integrating out the hard scale, $m_\mu \sim m_\pi$

HBET Feynman diagrams ←

2) Matching NRQED to pNRQED. Integrating out the soft scale, $m_\mu v$

Potential = Wilson loops = HQET-like Feynman diagrams ←

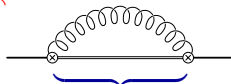
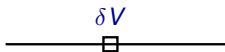
3) Observable: Spectrum or decays

Corrections to the Green Function ($h_s^{(0)} = \mathbf{p}^2/m + V_s^{(0)}$)

$$G_s(E) = P_s \frac{1}{h_s^{(0)} - H_I - E} P_s = G_s^{(0)} + \delta G_s \quad G_s^{(0)}(E) = \frac{1}{h_s^{(0)} - E}$$

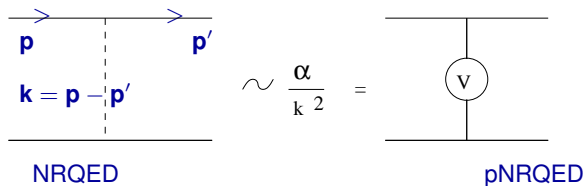
A) Ultrasoft loops (lamb shift-like): $\mathbf{x} \cdot \mathbf{E}$ ←

B) Quantum mechanics perturbation theory ←



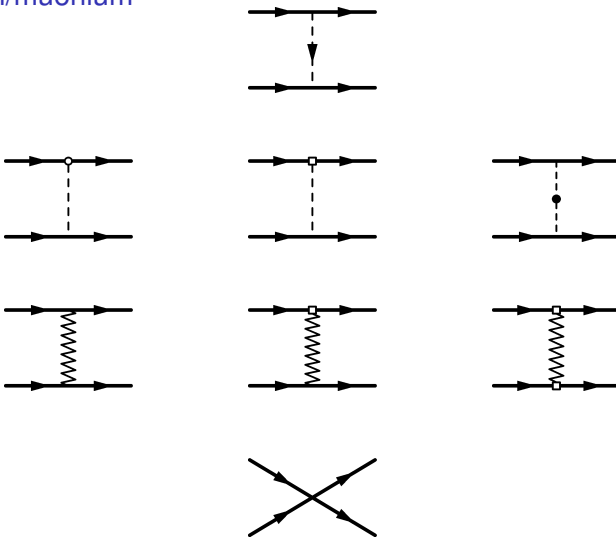
$$1/(E - V_s^{(0)} - \mathbf{p}^2/m)$$

Matching NRQED to pNRQED



Positronium/muonium

Tree level



Order $1/m^2$

$$\tilde{V}^{(b)} = \frac{\pi\alpha}{2} \left[Z_p \frac{c_D^{(\mu)}}{m_\mu^2} + Z_\mu \frac{c_D^{(p)}}{m_p^2} \right],$$

$$\tilde{V}^{(c)} = -i2\pi\alpha \frac{(\mathbf{p} \times \mathbf{k})}{\mathbf{k}^2} \cdot \left\{ Z_p \frac{c_S^{(\mu)} \mathbf{s}_1}{m_\mu^2} + Z_\mu \frac{c_S^{(p)} \mathbf{s}_2}{m_p^2} \right\},$$

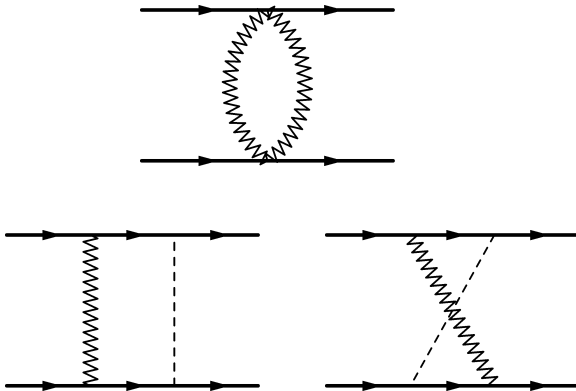
$$\tilde{V}^{(d)} = -Z_\mu Z_p 16\pi\alpha \left(\frac{d_2^{(\mu)}}{m_\mu^2} + \frac{d_2^{(\tau)}}{m_\tau^2} + \frac{d_{2,NR}}{m_p^2} \right),$$

$$\tilde{V}^{(e)} = -Z_\mu Z_p \frac{4\pi\alpha}{m_\mu m_p} \left(\frac{\mathbf{p}^2}{\mathbf{k}^2} - \frac{(\mathbf{p} \cdot \mathbf{k})^2}{\mathbf{k}^4} \right),$$

$$\tilde{V}^{(f)} = -\frac{i4\pi\alpha}{m_\mu m_p} \frac{(\mathbf{p} \times \mathbf{k})}{\mathbf{k}^2} \cdot (Z_p c_F^{(\mu)} \mathbf{s}_1 + Z_\mu c_F^{(p)} \mathbf{s}_2),$$

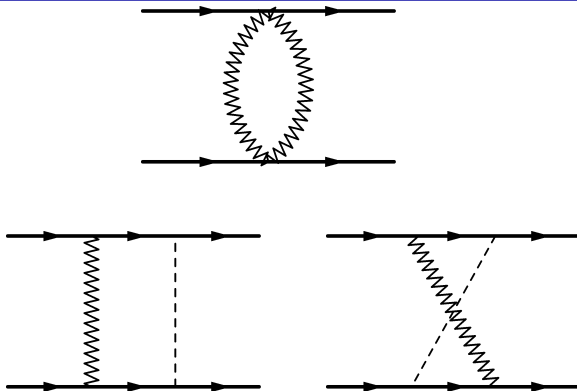
$$\tilde{V}^{(g)} = \frac{4\pi\alpha c_F^{(1)} c_F^{(2)}}{m_\mu m_p} \left(\mathbf{s}_1 \cdot \mathbf{s}_2 - \frac{\mathbf{s}_1 \cdot \mathbf{k} \mathbf{s}_2 \cdot \mathbf{k}}{\mathbf{k}^2} \right),$$

$$\tilde{V}^{(h)} = -\frac{1}{m_p^2} \left\{ (c_3^{pl_i} + 3c_4^{pl_i}) - 2c_4^{pl_i} \mathbf{S}^2 \right\}.$$



$$\tilde{V}_{1loop}^{(a)} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p} \left(\log \frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3} \log 2 + \frac{5}{3} \right),$$

$$\tilde{V}_{1loop}^{(b,c)} = \frac{4Z_\mu^2 Z_p^2 \alpha^2}{3m_\mu m_p} \left(\log \frac{\mathbf{k}^2}{\mu^2} + 2 \log 2 - 1 \right).$$



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Muonic Hydrogen: electron vacuum polarization

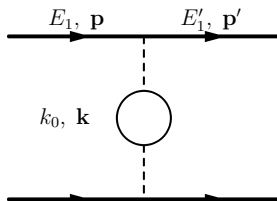


Figure: Leading correction to the Coulomb potential due to the electron vacuum polarization. $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ and $k_0 = E_1 - E_1'$.

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_\rho \alpha_V(k) \frac{1}{\mathbf{k}^2},$$

$$\alpha_{\text{eff}}(k) = \alpha \frac{1}{1 + \Pi(-\mathbf{k}^2)},$$

where

$$\Pi(k^2) = \alpha \Pi^{(1)}(k^2) + \alpha^2 \Pi^{(2)}(k^2) + \alpha^3 \Pi^{(3)}(k^2) + \dots$$

$$\alpha_V(k) = \alpha_{\text{eff}}(k) + \sum_{\substack{n,m=0 \\ n+m=\text{even}>0}} Z_\mu^n Z_\rho^m \alpha_{\text{eff}}^{(n,m)}(k) = \alpha_{\text{eff}}(k) + \delta\alpha(k), \quad \delta\alpha(k) = O(\alpha^4)$$

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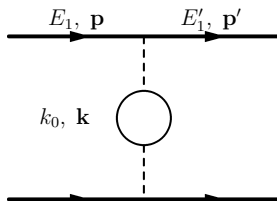


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Order $1/m^2$

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Order $1/m^2$ from energy-dependent terms

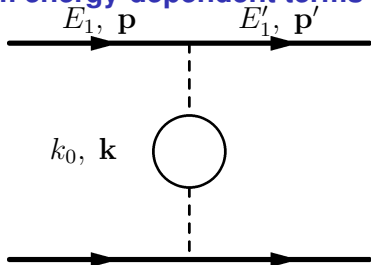


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$$u(q^2) = \frac{1}{3} \sqrt{1 - \frac{4}{q^2}} \left(1 + \frac{2}{q^2}\right).$$

Theoretical setup (muonic hydrogen Lamb shift:

$$\Delta E_L \equiv E(2P_{3/2}) - E(2S_{1/2}))$$

$$L_{pNRQED} = \int d^3\mathbf{r} d^3\mathbf{R} dt S^\dagger(\mathbf{r}, \mathbf{R}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} \right. \\ \left. - V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + e\mathbf{r} \cdot \mathbf{E}(\mathbf{R}, t) \right\} S(\mathbf{r}, \mathbf{R}, t) - \int d^3\mathbf{r} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

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Vacuum polarization effects: $\mathcal{O}(m_r\alpha^3)$

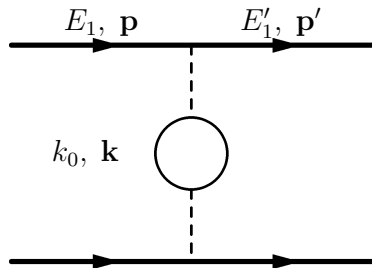
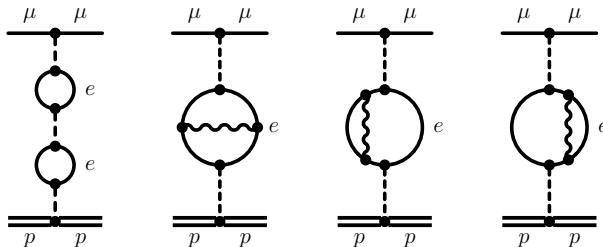


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1-loop static potential

$$E_{LO} = \langle n | \delta V | n \rangle = 205.0074 \text{ meV} = \mathcal{O}(m_r\alpha^3)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



Pachuki/Borie

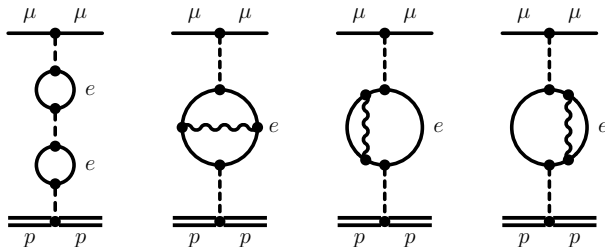
2-loop static potential is the same as two-loop vacuum polarization iterations (*two loop vacuum polarization*)

$$\delta E = \langle n | \delta V | n \rangle = 1.5079 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Quantum mechanics perturbation theory (*iteration one-loop*)

$$\delta E \sim \langle n | \delta V \frac{1}{H_C - E_n} \delta V | n \rangle = 0.151 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

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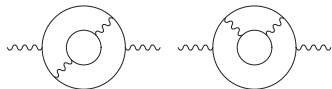
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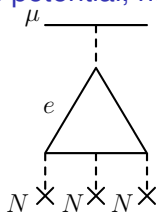
3-loop static potential (three loop vacuum polarization, Kinoshita-Nio)

$$0.00752 \text{ meV} = \mathcal{O}(m_r\alpha^5)$$

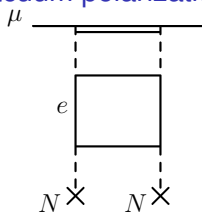
Slightly corrected by Ivanov et al.



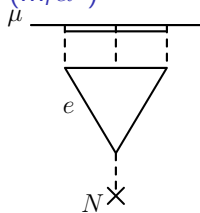
Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



(1:3)



(2:2)



(3:1)

Light-by-light (Wichmann-Kroll and Delbrück) contribution very small (Karshenboim et al.)

$$\Delta E \simeq -0.0009 \text{ meV} = \mathcal{O}(m_r\alpha^5)$$

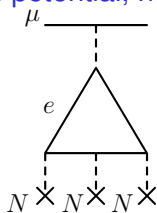
Earlier work by Borie

Observation:

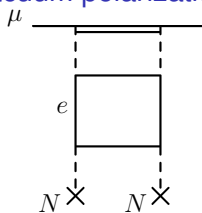
The limit $m_e \rightarrow 0$ known from QCD (Anzai et al. and Smirnov et al).

It should be possible to obtain the result with finite mass (albeit numerically) and check.

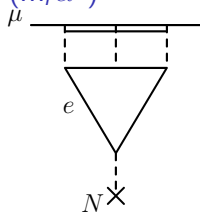
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$1/m$ potential

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(1)}(r)}{m_\mu} \rightarrow \mathcal{O}(m_r \alpha^6)$$

relativistic corrections+vacuum polarization

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$$\frac{V^{(2)}(r)}{m_\mu^2} \rightarrow \mathcal{O}(m_r \alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

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$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

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$\mathcal{O}(m\alpha^5)$ 0.0169 (Pachucki and Veitia; Borie)

relativistic corrections+vacuum polarization

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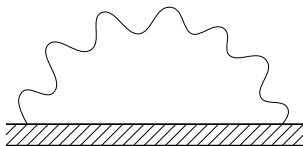
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$\mathcal{O}(m\alpha^5)$ 0.018759 (Jentschura; Karshenboim&Ivanov&Korzinin; Peset&Pineda)

Ultrasoft effects: $\mathcal{O}(m\alpha^5)$



$$\Delta E = -0.6677 \text{ meV}$$

$$\mathcal{O}(m\alpha^5 \frac{m_\mu}{m_p}) : \quad \Delta E = -0.045 \text{ meV}$$

All (soft+ultrasoft):

$$\Delta E = -0.71896 \text{ meV.}$$

Start the overlap with hadronic effects.

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

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$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.} \delta^3(\mathbf{r}) \rightarrow \Delta E \sim \frac{1}{m_p^2} D_d^{had.} (m_r \alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D$$

c_3, d_2, c_D, \dots matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \dots \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \dots - e \frac{c_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p + \dots + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

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HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

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$$\mathcal{L}_{N,l} = \frac{1}{m_p^2} \sum_i c_{3,R}^{pl_i} \bar{N}_p \gamma^0 N_p \bar{l}_i \gamma^0 l_i + \frac{1}{m_p^2} \sum_i c_{4,R}^{pl_i} \bar{N}_p \gamma^j N_p \bar{l}_i \gamma_j l_i$$

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HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \dots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\text{Tr}[D_\mu U D^\mu U] + \dots \quad U = u^2 = e^{i\frac{\pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger (iv^\mu \nabla_\mu + g_A u_\mu S^\mu) N + \dots + (\Delta) + \dots - e\frac{c_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \quad \nabla_\mu = \partial_\mu + \Gamma_\mu \quad u_\mu = iu^\dagger (\nabla_\mu U) u$$

$$\Gamma_\mu = \frac{1}{2} \left\{ u^\dagger (\partial_\mu + ieQA_\mu) u + u (\partial_\mu + ieQA_\mu) u^\dagger \right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2} \sum_i c_{3,R}^{pl_i} \bar{N}_p \gamma^0 N_p \bar{l}_i \gamma^0 l_i + \frac{1}{m_p^2} \sum_i c_{4,R}^{pl_i} \bar{N}_p \gamma^j N_p \bar{l}_i \gamma_j l_i$$

$$\delta\mathcal{L} = \dots \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \dots - e\frac{c_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p + \dots + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

Hadronic vacuum polarization effects

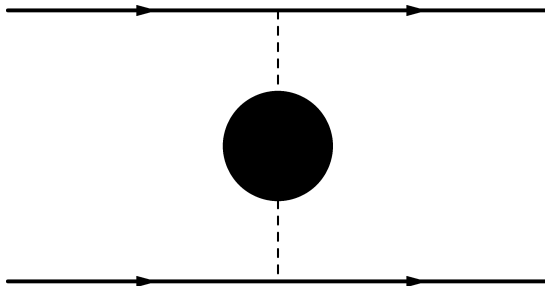


Figure: *Leading correction to the Coulomb potential due to the hadronic vacuum polarization.*

$d_2 \rightarrow$ hadronic vacuum polarization

$$\Delta E = 0.011 \text{ meV}$$

Hadronic vacuum polarization effects

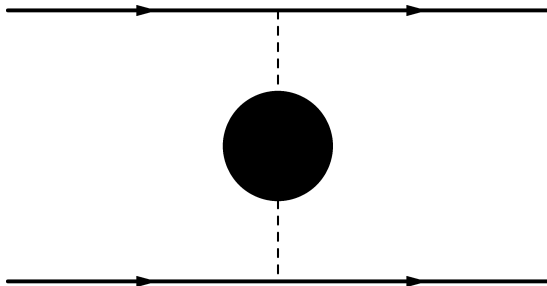
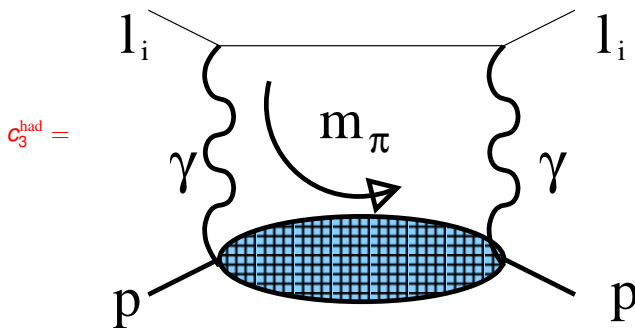


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$d_2 \rightarrow$ hadronic vacuum polarization

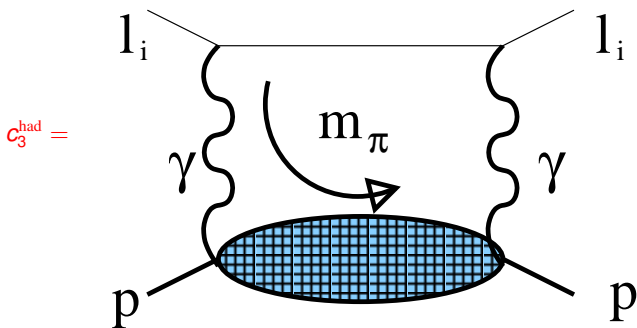
$$\Delta E = 0.011 \text{ meV}$$



$$T^{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle p, s | T J^\mu(x) J^\nu(0) | p, s \rangle,$$

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) + \frac{1}{m_p^2} \left(p^\mu - \frac{m_p \rho}{q^2} q^\mu \right) \left(p^\nu - \frac{m_p \rho}{q^2} q^\nu \right) S_2(\rho, q^2)$$

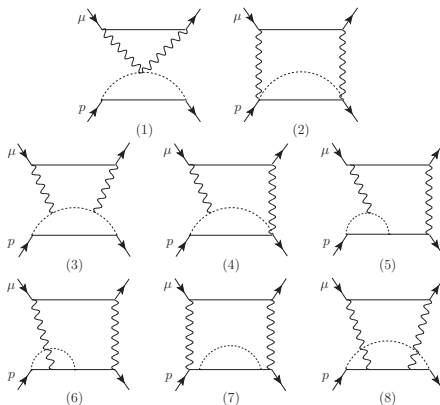
$$S_1 = ?? \quad S_2 = ??$$



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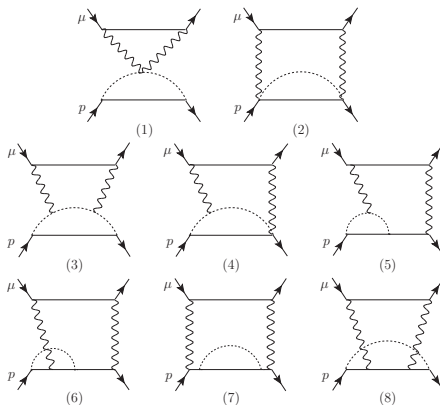


m_μ extra suppression+ χ PT (Model independent)

Power-like chiral enhanced ($\rightarrow \chi$ PT can predict the leading order!)

$$c_3^{\text{had}} \sim \alpha^2 \frac{m_\mu}{m_\pi} + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) \quad \delta E \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$$

$$\text{Error: LO} \times \frac{m_\pi}{\Delta} \simeq \text{LO} \times \frac{1}{2} \rightarrow c_3^{\text{had}} = \alpha^2 \frac{m_\mu}{m_\pi} 47.2(23.6)$$

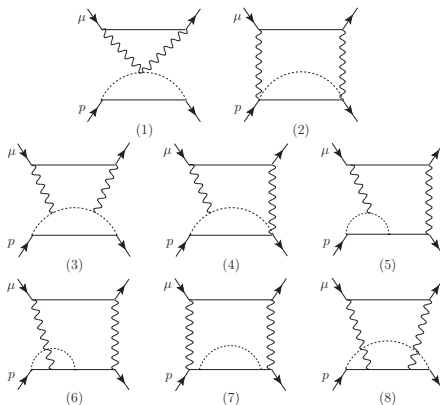


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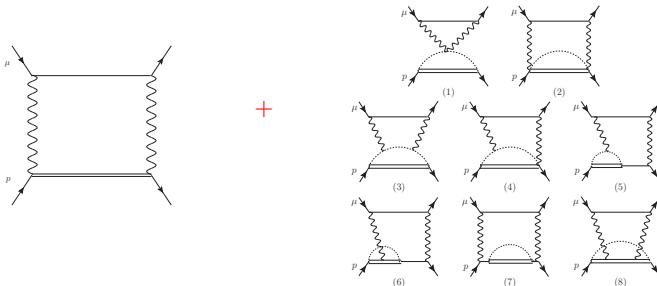
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Large N_c . Including the Δ particle

Error:

$$\frac{m_\mu}{\Delta} \sim N_c \frac{m_\mu}{\Lambda_{QCD}} \rightarrow \frac{m_\mu}{\Lambda_{QCD}} \sim \frac{1}{3}$$



$$c_3^{\text{had}} \sim \alpha^2 \frac{m_\mu}{m_\pi} \left[1 + \# \frac{m_\pi}{\Delta} + \dots \right] + \mathcal{O} \left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}} \right) = \alpha^2 \frac{m_\mu}{m_\pi} \begin{cases} 47.2(23.6) & (\pi), \\ 56.7(20.6) & (\pi + \Delta), \end{cases}$$

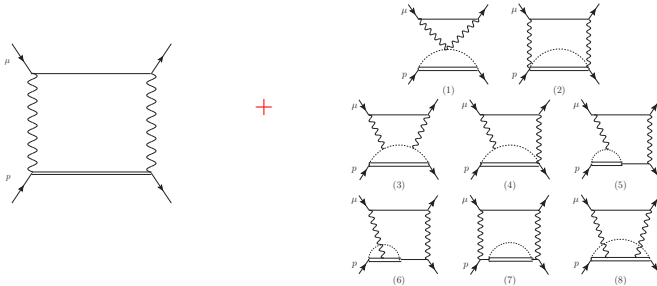
$$\Delta E_{\text{TPE}} = 28.59(\pi) + 5.86(\pi \& \Delta) = 34.4(12.5) \mu\text{eV} \quad (\text{Peset\&AP}).$$

(Model dependent: $\Delta E_{\text{TPE}} = 33(2) \mu\text{eV}$ (Birse-McGovern))

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(Model dependent: $\Delta E_{\text{TPE}} = 33(2) \mu\text{eV}$ (Birse-McGovern))

C_3^{had} : Two-Photon-Exchange contribution = Born + polarizability

Born:

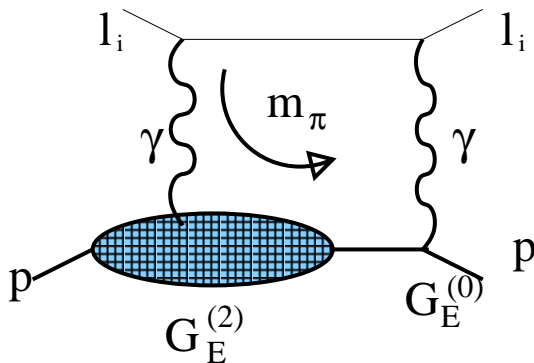


Figure: Symbolic representation (plus permutations) of the Born $\langle r^3 \rangle$ correction.

$$\Delta E_{\text{Born}} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3} = \frac{48}{\pi} \int \frac{d^3 k}{4\pi} \frac{1}{\mathbf{k}^6} \left(G_E^2 - 1 + \frac{1}{3} \langle r^2 \rangle \mathbf{k}^2 \right) = \frac{96}{\pi} \int \frac{d^{D-1} k}{4\pi} \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\Delta E_{\text{Born}} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\begin{aligned} \delta C_{3,\text{Born}}^{pl} &= \frac{\pi}{3} \alpha^2 m_p^2 m_\mu \langle r^3 \rangle_{(2)} = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0} \right)^2 \frac{m_l}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} \right. \\ &\quad \left. + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta} \right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta} \right)^{2r} \right\}, \end{aligned}$$

where $(\Delta = M_\Delta - M_p \sim 300 \text{ MeV})$

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \quad r \geq 0,$$

$$B_n \equiv \int_0^\infty dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln \left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1} \right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

$$\Delta E_{\text{Born}} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\begin{aligned} \delta C_{3,\text{Born}}^{pl_i} &= \frac{\pi}{3} \alpha^2 m_p^2 m_\mu \langle r^3 \rangle_{(2)} = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0} \right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} \right. \\ &\quad \left. + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta} \right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta} \right)^{2r} \right\}, \end{aligned}$$

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$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

	$\langle r^3 \rangle$	$\langle r^5 \rangle$	$\langle r^6 \rangle$	$\langle r^3 \rangle_{(2)}$
π	0.4980	1.619	5.203	0.9960
$\pi \& \Delta$	0.4071	1.522	4.978	0.8142
<i>Dipole</i>	0.7706	1.775	3.325	2.023
<i>Kelly</i>	0.9838	3.209	7.440	2.526
<i>Distler et al.</i>	1.16(4)	8.0(1.2)(1.0)	29.8(7.6)(12.6)	2.85(8)

Table: The first two rows give the prediction from the effective theory (Peset&AP). The third row corresponds to the standard dipole fit with $\langle r^2 \rangle = 0.6581 \text{ fm}^2$. The fourth and fifth rows correspond to different parameterizations of experimental data. For completeness, we also quote $\langle r^3 \rangle_{(2)} = 2.71 \text{ fm}^3$ from Friar.

μeV	DR	<i>Pachucki</i>	<i>Carlson et al</i>	HBET	<i>Peset&AP</i> (π)	($\pi \& \Delta$)
ΔE_{Born}		23.2(1.0)	24.7(1.6)		10.1(5.1)	8.3(4.3)

Table: Predictions for the Born contribution to the $n = 2$ Lamb shift. The first two entries correspond to dispersion relations. The last two entries are the predictions of HBET: The 3rd entry is the prediction of HBET at leading order (only pions) and the last entry is the prediction of HBET at leading and next-to-leading order (pions and Deltas).

c_3 Polarizability effects

(μeV)	[1]	[2]	[3]	[4]	$B_{\chi\text{PT}}(\pi)$	HBET(π)	($\pi\&\Delta$)
ΔE_{pol}	12(2)	11.5	7.4(2.4)	15.3(5.6)	8.2($^{+1.2}_{-2.5}$)	18.5(9.3)	26.2(10.0)

Table: *Polarizability contribution to the $n = 2$ Lamb shift. The first four entries use dispersion relations for the inelastic term and different modeling functions for the subtraction term. [1] Pachucki, [2] Martynenko, [3] Carlson&Vanderhaeghen, [4] Gorchtein et al.. The 5th entry is the prediction obtained using $B_{\chi\text{PT}}$ (Alarcon et al.). The last two entries are the predictions of HBET (Nevado&AP and Peset&AP).*

Polarizability=Inelastic+subtraction

$$c_{3,\text{sub}}^{p_i} = -e^4 M_p m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} (3k_{0,E}^2 + \mathbf{k}^2) S_1(0, -k_E^2)$$

$$c_{3,\text{inel}}^{p_i} = -e^4 M_p m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \\ \times \left\{ (3k_{0,E}^2 + \mathbf{k}^2) (S_1(ik_{0,E}, -k_E^2) - S_1(0, -k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\}$$

$$\Delta E^{(\text{sub})}(\pi\text{-loop}) = -1.62 \mu\text{eV}; \quad \Delta E^{(\text{sub})}(\pi\Delta\text{-loop}) = -1.23 \mu\text{eV}.$$

$$\delta c_{3,\text{sub}}^{p_i} \sim -\delta c_{3,\text{inel}}^{p_i} \simeq -\frac{4}{3} \alpha^2 \frac{m_{l_i}}{\Delta} b_{1,F}^2 \ln(\nu/m_{l_i}) \rightarrow \Delta E^{(\text{sub})}|_{\nu=m_p} \sim -11.4 \mu\text{eV}$$

Polarizability=Inelastic+subtraction

$$\mathcal{C}_{3,\text{sub}}^{pl_i} = -e^4 M_p m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} (3k_{0,E}^2 + \mathbf{k}^2) S_1(0, -k_E^2)$$

$$\begin{aligned} \mathcal{C}_{3,\text{inel}}^{pl_i} &= -e^4 M_p m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \\ &\times \left\{ (3k_{0,E}^2 + \mathbf{k}^2) (S_1(ik_{0,E}, -k_E^2) - S_1(0, -k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\} \end{aligned}$$

$$\Delta E^{(\text{sub})}(\pi\text{-loop}) = -1.62 \mu\text{eV}; \quad \Delta E^{(\text{sub})}(\pi\Delta\text{-loop}) = -1.23 \mu\text{eV}.$$

$$\delta\mathcal{C}_{3,\text{sub}}^{pl_i} \sim -\delta\mathcal{C}_{3,\text{inel}}^{pl_i} \simeq -\frac{4}{3}\alpha^2 \frac{m_{l_i}}{\Delta} b_{1,F}^2 \ln(\nu/m_{l_i}) \rightarrow \Delta E^{(\text{sub})}|_{\nu=m_p} \sim -11.4 \mu\text{eV}$$

Definition of the proton radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \dots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2} F_2(q^2), \quad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6 \frac{d}{dq^2} G_{E,p}(q^2) |_{q^2=0}$$

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Definition of the proton radius

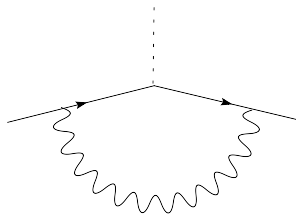
$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \dots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2} F_2(q^2), \quad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2(\nu) = 6 \frac{d}{dq^2} G_{E,p}(q^2) |_{q^2=0}$$

Infrared divergent! \rightarrow Wilson coefficient



Definition of the proton radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \dots$$

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$$r_p^2(\nu) = 6 \frac{d}{dq^2} G_{E,p}(q^2) \Big|_{q^2=0} = \frac{3}{4} \frac{1}{m_p^2} \left(c_D^{(p)}(\nu) - 1 \right)$$

$$c_D(\nu) = 1 + 2F_2 + 8F_2' = 1 + 8m_p^2 \frac{dG_{p,E}(q^2)}{dq^2} \Big|_{q^2=0},$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4} \frac{1}{m_p^2} (c_D(\nu) - c_{D,\text{point-like}}(\nu))$$

$$c_{D,\text{point-like}} = 1 + \frac{\alpha}{\pi} \left(\frac{4}{3} \ln \frac{m_p^2}{\nu^2} \right)$$

Definition of the proton radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \dots$$

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$$c_D(\nu) = 1 + 2F_2 + 8F_2' = 1 + 8m_p^2 \frac{dG_{p,E}(q^2)}{dq^2} \Big|_{q^2=0},$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4} \frac{1}{m_p^2} (c_D(\nu) - c_{D,\text{point-like}}(\nu))$$

$$c_{D,\text{point-like}} = 1 + \frac{\alpha}{\pi} \left(\frac{4}{3} \ln \frac{m_p^2}{\nu^2} \right)$$

Definition of the proton radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \dots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2} F_2(q^2), \quad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2(\nu) = 6 \frac{d}{dq^2} G_{E,p}(q^2) \Big|_{q^2=0} = \frac{3}{4} \frac{1}{m_p^2} \left(c_D^{(p)}(\nu) - 1 \right)$$

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Definition of the neutron radius

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$$F_i(q^2) = 0 + \frac{q^2}{m_p^2} F_i' + \dots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2} F_2(q^2), \quad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_n^2 = 6 \frac{d}{dq^2} G_{n,E}(q^2) \Big|_{q^2=0} = \frac{3}{4} \frac{1}{m_p^2} c_D^{(n)}$$

$$c_D = 0 + 2F_2 + 8F_1' = 0 + 8m_n^2 \frac{dG_{n,E}(q^2)}{dq^2} \Big|_{q^2=0}$$

Standard definition (corresponds to the experimental number):

$$r_n^2 = \frac{3}{4} \frac{1}{m_n^2} c_D$$

Neutron-lepton scattering length = REAL low energy constant

$$b_{nl} = \frac{1}{4m_n} \left(\alpha c_D - \frac{2}{\pi} c_{3,NR}^{nl} \right) \sim D_d^{(n)had}$$

It is **not** proportional to the radius

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	$\mathcal{O}(m_r \alpha^3)$	$V_{VP}^{(0)}$	205. 00737
	$\mathcal{O}(m_r \alpha^4)$	$V_{VP}^{(0)}$	1. 50795
	$\mathcal{O}(m_r \alpha^4)$	$V_{VP}^{(0)}$	0. 15090
	$\mathcal{O}(m_r \alpha^5)$	$V_{VP}^{(0)}$	0. 00752
	$\mathcal{O}(m_r \alpha^5)$	$V_{LbL}^{(0)}$	-0. 00089(2)
	$\mathcal{O}(m_r \alpha^4 \times \frac{m_\mu^2}{m_p^2})$	$V^{(2,1)} + V^{(3,0)}$	0. 05747
	$\mathcal{O}(m_r \alpha^5)$	$V_{VP}^{(2,2)} + V^{(2,1)} \times V_{VP}^{(0,2)}$	0. 01876
	$\mathcal{O}(m_r \alpha^5)$	$V_{no-VP}^{(2,2)} + \text{ultrasoft}$	-0. 71896
	$\mathcal{O}(m_r \alpha^6 \times \ln(\frac{m_\mu}{m_e}))$	$V^{(2,3)}; c_D^{(\mu)}$	-0. 00127
	$\mathcal{O}(m_r \alpha^6 \times \ln \alpha)$	$V_{VP}^{(2,3)}; c_D^{(\mu)}$	-0. 00454
	$\mathcal{O}(m_r \alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}; c_D^{(\rho)}$	-5. 1975 $\frac{r_p^2}{\text{fm}^2}$
	$\mathcal{O}(m_r \alpha^5 \times m_r^2 r_p^2)$	$V_{VP}^{(2,2)} + V^{(2,1)} \times V_{VP}^{(0,2)}; c_D^{(\rho)}$	-0. 0282 $\frac{r_p^2}{\text{fm}^2}$
	$\mathcal{O}(m_r \alpha^6 \ln \alpha \times m_r^2 r_p^2)$	$V^{(2,3)}; c_D^{(\rho)}$	-0. 0014 $\frac{r_p^2}{\text{fm}^2}$
	$\mathcal{O}(m_r \alpha^5 \times \frac{m_r^2}{m_p^2})$	$V_{VP}^{(2)}; d_2^{\text{had}}$	0. 0111(2)
	$\mathcal{O}(m_r \alpha^5 \times \frac{m_r^2 m_\mu}{m_p^2 m_\pi})$	$V^{(2)}; c_3^{\text{had}}$	0. 0344(125)

$$\Delta E_L^{\text{our work}} = \left[206.0243(30) - 5.2270(7) \frac{r_p^2}{\text{fm}^2} + 0.0455(125) \right] \text{ meV} .$$

This expression includes the leading logarithmic $\mathcal{O}(m_\mu \alpha^6)$ terms, as well as the leading $\mathcal{O}\left(m_\mu \alpha^5 \frac{m_\mu^2}{m_\rho^2}\right)$ hadronic effects. The accuracy of our result is limited by uncomputed terms of $\mathcal{O}(m_\mu \alpha^5 \frac{m_\mu^3}{m_\rho^3}, m_\mu \alpha^6)$.

Using

$$\Delta E_L^{\text{exp}} \equiv E(2P_{3/2}) - E(2S_{1/2}) = 202.3706(23) \text{ meV} \quad \text{Antognini et al.}$$

$$r_p = 0.8413(15) \text{ fm.}$$

At 6.8σ variance with respect the CODATA value.

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Hadronic corrections: Spin-dependent

Pineda: hep-ph/0210210, hep-ph/0308193; Peset&Pineda, 1406.4524

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.} (\mathbf{S}_1 + \mathbf{S}_2)^2 \delta^3(r)$$

$$D_s^{had.} = 2c_4$$

c_4 , matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \sigma N_p \mu^\dagger \sigma \mu$$

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$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.} (\mathbf{S}_1 + \mathbf{S}_2)^2 \delta^3(r)$$

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c_4 , Spin-dependent effects

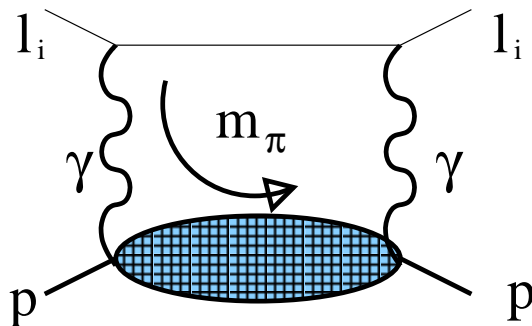


Figure: Symbolic representation (plus permutations) of the spin-dependent correction.

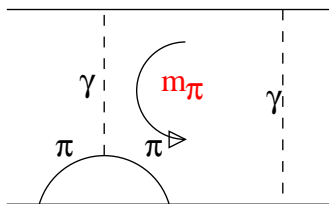
$$c_4^{pl} = -\frac{ig^4}{3} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{1}{k^4 - 4m_i^2 k_0^2} \left\{ A_1(k_0, k^2)(k_0^2 + 2k^2) + 3k^2 \frac{k_0}{m_p} A_2(k_0, k^2) \right\}$$

$$T^{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle p, s | T J^\mu(x) J^\nu(0) | p, s \rangle ,$$

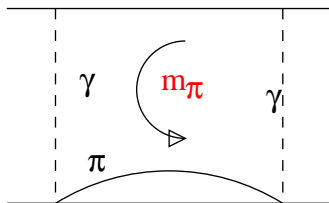
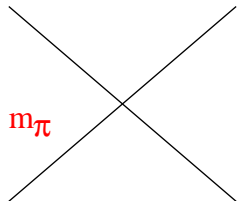
which has the following structure ($\rho = q \cdot p/m$):

$$\begin{aligned} T^{\mu\nu} = & \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) \\ & + \frac{1}{m_p^2} \left(p^\mu - \frac{m_p \rho}{q^2} q^\mu \right) \left(p^\nu - \frac{m_p \rho}{q^2} q^\nu \right) S_2(\rho, q^2) \\ & - \frac{i}{m_p} \epsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\rho, q^2) \\ & - \frac{i}{m_p^3} \epsilon^{\mu\nu\rho\sigma} q_\rho \left((m_p \rho) s_\sigma - (q \cdot s) p_\sigma \right) A_2(\rho, q^2) \end{aligned}$$

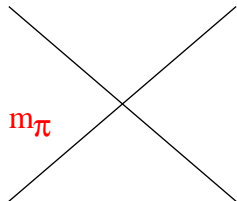
Leading chiral logs to the hyperfine splitting



$$\sim \frac{1}{f_\pi^2} \ln m_\pi$$



$$\sim \frac{1}{f_\pi^2} \ln m_\pi$$



$$\delta V = 2 \frac{C_4}{m_p^2} \mathbf{S}^2 \delta^{(3)}(\mathbf{r}).$$

$$\delta E_{HF} \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$$

The leading chiral logs are determined for **Hydrogen** and **muonic hydrogen** hyperfine splitting. Model independent result!

$$\begin{aligned} C_4^{p_i} &\simeq \left(1 - \frac{\mu_p^2}{4}\right) \alpha^2 \ln \frac{m_i^2}{\nu^2} \\ &+ \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \frac{2}{3} \left(\frac{2}{3} + \frac{7}{2\pi^2}\right) \pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2} \\ &+ \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \frac{8}{27} \left(\frac{5}{3} - \frac{7}{\pi^2}\right) \pi^2 g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \\ &\stackrel{(N_c \rightarrow \infty)}{\simeq} \alpha^2 \ln \frac{m_i^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2}. \end{aligned}$$

$$E_{HF} = 4 \frac{C_4^{p_i}}{m_p^2} \frac{1}{\pi} (\mu_{i,p} \alpha)^3 \sim m_i \alpha^5 \frac{m_i^2}{m_p^2} \times (\ln m_q, \ln \Delta, \ln m_i).$$

$$C_4^{plj} = C_{4,R}^{plj} + C_{4,\text{point-like}}^{plj} + C_{4,\text{Born}}^{plj} + C_{4,\text{pol}}^{plj} + \mathcal{O}(\alpha^3).$$

$$C_{4,\text{point-like}}^{plj} = \left(1 - \frac{\kappa_p^2}{4}\right) \alpha^2 \ln \frac{m_{l_i}^2}{\nu^2},$$

$$C_{4,\text{Born}}^{plj} \simeq (4\pi\alpha)^2 M_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(1)}$$

$$\simeq \frac{M_p^2}{(4\pi F_0)^2} \alpha^2 \frac{2}{3} \pi^2 \left[g_A^2 \ln \frac{m_\pi^2}{\nu^2} + \frac{4}{9} g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \right],$$

$$C_{4,\text{pol}}^{plj} = \frac{M_p^2}{(4\pi F_0)^2} \frac{\alpha^2}{\pi} \frac{8}{3} \left(\frac{7\pi}{8} - \frac{\pi^3}{12} \right) \left[g_A^2 \ln \frac{m_\pi^2}{\nu^2} - \frac{8}{9} g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \right]$$

$$+ \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2}.$$

$$C_{4,\text{point-like}}^{plj} + C_{4,\text{pol}}^{plj} \stackrel{(N_c \rightarrow \infty)}{\simeq} \alpha^2 \ln \frac{m_{l_i}^2}{\nu^2}$$

Polarizability contribution small and vanishes in the large N_c limit!!

$$c_4^{plj} = c_{4,R}^{plj} + c_{4,\text{point-like}}^{plj} + c_{4,\text{Born}}^{plj} + c_{4,\text{pol}}^{plj} + \mathcal{O}(\alpha^3).$$

$$c_{4,\text{point-like}}^{plj} = \left(1 - \frac{\kappa_p^2}{4}\right) \alpha^2 \ln \frac{m_{l_i}^2}{\nu^2},$$

$$\begin{aligned} c_{4,\text{Born}}^{plj} &\simeq (4\pi\alpha)^2 M_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(1)} \\ &\simeq \frac{M_p^2}{(4\pi F_0)^2} \alpha^2 \frac{2}{3} \pi^2 \left[g_A^2 \ln \frac{m_\pi^2}{\nu^2} + \frac{4}{9} g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \right], \end{aligned}$$

$$\begin{aligned} c_{4,\text{pol}}^{plj} &= \frac{M_p^2}{(4\pi F_0)^2} \frac{\alpha^2}{\pi} \frac{8}{3} \left(\frac{7\pi}{8} - \frac{\pi^3}{12} \right) \left[g_A^2 \ln \frac{m_\pi^2}{\nu^2} - \frac{8}{9} g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \right] \\ &\quad + \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2}. \end{aligned}$$

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$$c_{4,\text{point-like}}^{plj} + c_{4,\text{pol}}^{plj} \stackrel{(N_c \rightarrow \infty)}{\simeq} \alpha^2 \ln \frac{m_{l_j}^2}{\nu^2}$$

Polarizability contribution small and vanishes in the large N_c limit!!

c_4 , Spin-dependent effects (Born): $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$

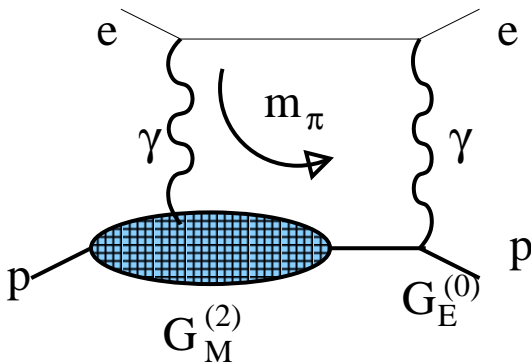


Figure: *Symbolic representation (plus permutations) of the Born correction.*

$$c_{4,\text{Born}}^{pl} = (4\pi\alpha)^2 M_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{k^4} G_E^{(0)} G_M^{(2)}.$$

Zemach magnetic radius

Chiral logs can be determined and constitute the leading contribution!

$$\langle r_Z \rangle = -\frac{4}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left[G_E(Q^2) G_M(Q^2) - 1 \right] = -\frac{4}{\pi} \int_0^\infty q^{D-4} \frac{dQ}{Q^2} G_E^{(0)}(Q^2) G_M^{(2)}(Q^2).$$

$$\langle r_Z \rangle = -\frac{3}{4\pi} \frac{1}{\alpha^2 M_p} C_{4,\text{Born}}^{p_i} \simeq -\frac{\pi}{2} \frac{M_p}{(4\pi F_0)^2} \left[g_A^2 \ln \frac{m_\pi^2}{\nu^2} + \frac{4}{9} g_{\pi N \Delta}^2 \ln \frac{\Delta^2}{\nu^2} \right] \stackrel{(\nu=m_\rho)}{=} 1.35 \text{ fm}.$$

"Experiment" $\sim 1.04 - 1.08 \text{ fm}$.

Hydrogen. Fixing $\nu = m_p$ we obtain:

$$E_{\text{HF,logarithms}}(m_p) = -0.031 \text{ MHz}.$$

It accounts for 2/3 of the difference between theory (QED) and experiment.

$$E_{\text{HF}}(\text{QED}) - E_{\text{HF}}(\text{exp}) = -0.046 \text{ MHz}.$$

What is left gives the size of the counterterm. Experimentally what we have is $c_4^{p/e} = -48\alpha^2$ and $c_{4,R}^{p/l}(m_p) \simeq c_{4,R}^p(m_p) \simeq -16\alpha^2$.

Muonic hydrogen.

$$c_4^{p/l\mu} = c_4^{p/e} + [c_{4,\text{point-like}}^{p/l\mu} - c_{4,\text{point-like}}^{p/e}] + [c_{4,\text{pol}}^{p/l\mu} - c_{4,\text{pol}}^{p/e}] + \mathcal{O}(\alpha^3, \alpha^2 m_\mu / \Lambda_{\text{QCD}}).$$

$$c_{4,\text{point-like}}^{p/l\mu} - c_{4,\text{point-like}}^{p/e} = \left(1 - \frac{\kappa_p^2}{4}\right) \alpha^2 \ln \frac{m_\mu^2}{m_e^2} \simeq 2.09\alpha^2$$

$$c_{4,\text{pol}}^{p/l\mu} - c_{4,\text{pol}}^{p/e} = 0.17\alpha^2(\pi) + 0.07\alpha^2(\Delta) + 0.008\alpha^2(\pi \& \Delta) = 0.24\alpha^2.$$

Overall we obtain $c_4^{p/l\mu} \simeq -46\alpha^2$. The bulk of this contribution comes from the Born term, which in turn is related to the Zemach magnetic radius

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Recent work:

Hagelstein et al.: 1511.04301&1512.03765

Nguyen Thu Huong et al: 1511.06255

Not clear if they are able to reproduce the HBET/leading chiral contribution (which is a model independent prediction).

CONCLUSIONS

Effective Field Theories provide with a model independent, efficient and systematic (Power counting) approach to the dynamics of NR systems and a unified framework to determine the nonperturbative effects.

Possible to obtain a rigorous connection between Quantum Field Theories and a NR Quantum-mechanical formulation of the NR systems.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

The two-photon exchange energy shift (and the associated error) is (and can only be) computed in a model independent way with χ PT. Overall number consistent with determinations from a combined use of dispersion relations and models, but individual contributions are quite different.

χ PT predicts the chiral logs of the hyperfine splitting and the difference between hydrogen and muonic hydrogen.

Analytic understanding of the QCD dynamics: m_q and N_c dependence.

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CONCLUSIONS

$$\Delta E_L^{\text{our work}} = \left[206.0243(30) - 5.2270(7) \frac{r_p^2}{\text{fm}^2} + 0.0455(125) \right] \text{ meV} .$$

Using

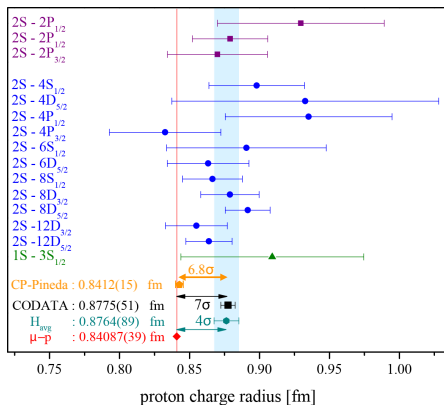
$$\Delta E_L^{\text{exp}} \equiv E(2P_{3/2}) - E(2S_{1/2}) = 202.3706(23) \text{ meV} \quad \text{Antognini et al.}$$

$$r_p = 0.8413(15) \text{ fm.}$$

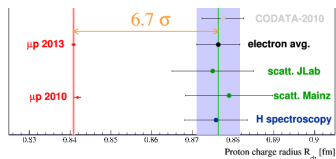
At 6.8σ variance with respect the CODATA value.

The proton radius in hydrogen spectroscopy

Experimental measurements



The proton radius in ep scattering



► Definition??

► very sensitive to low q^2 data:

extrapolation from $|\mathbf{q}| \geq 100$ MeV to $|\mathbf{q}| = 0$

$|\mathbf{q}| \sim m_e \sim 0.5$ MeV (muonic hydrogen)

$|\mathbf{q}| \sim m_e \alpha \sim 5 \cdot 10^{-3}$ MeV (hydrogen)

► dependence on the fitting functions: normalization factors, full data set ...

► Bonn group with dispersion relations:

$$r_p = 0.84^{+0.01}_{-0.01} \text{ fm.}$$