Precision calculations with nonrelativistic EFTs: proton radius from muonic hydrogen

(Potential NRQED: Matching (NR)QED and the Schroedinger equation)

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Intersections of BSM Phenomenology and QCD for New Physics Searches,
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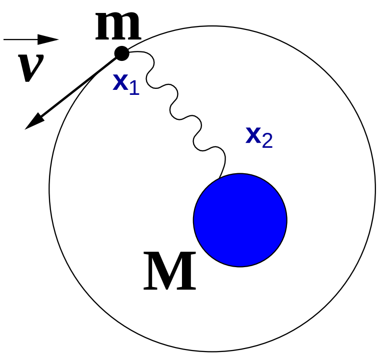
$v \sim \alpha \ll 1$



$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \qquad \mathbf{X} = \frac{m}{m+M}\mathbf{x}_1 + \frac{M}{m+M}\mathbf{x}_2$$

$$H = \frac{\mathbf{p}^2}{2m} + V(r) \qquad V(r) = -\frac{Z_1 Z_2 \alpha}{r}$$

Scales: hard, soft, ultrasoft; $m \gg mv \gg mv^2$...

Motivation

Real Motivation: to understand the connection between non-relativistic (NR) Quantum Mechanics and Quantum Field Theories.

"Physical Systems":

1) NR bound state systems:

- QED: positronium, Hydrogen-like/exotic atoms, atomic physics . . .
- QCD: Heavy Quarkonium (Υ, J/ψ, B_c . . .), . . .

2) Q-$\bar{Q}$ production near threshold (t-$\bar{t}$ at NLC).

3) Static systems $\leftrightarrow$ lattice "experimental" data.

Better understanding of QCD/QED, better determination of the parameters of the Standard Model: m_b, m_t, α_s, m_l, α_{em}, . . . , and hadronic low energy constants: r_p $\rightarrow$ Discrepancies may signal new physics

Tool: **Effective Field Theories** $\equiv$ **Factorization**

Why?: There is a hierarchy of different scales (hard, soft and ultrasoft).

$$m >> mv >> mv^2\,, \quad (\Lambda_{QCD})$$

EFTs are especially useful in these situations.

1) Perturbative calculations much easier and systematic.

2) Nonperturbative information is parameterized in a model independent way.

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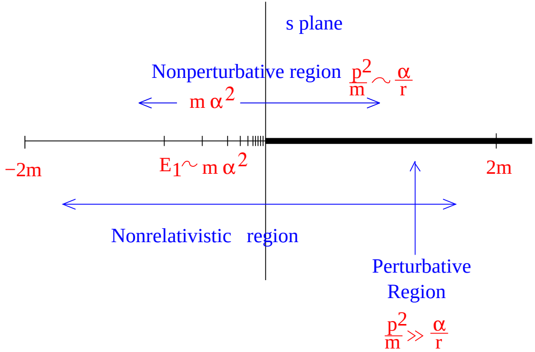
kinematical situation



1st approximation: ∞ number of NR (bound states) free particles
Unusual situation in EFTs. What we will get is somewhat unusual from the EFT point of view:

$$\mathcal{L} = \sum_n \psi_n^\dagger(\mathbf{X}, t)(i\partial_0 + \frac{\boldsymbol{\nabla}_X^2}{2M} - E_n + i\epsilon)\psi_n(\mathbf{X}, t)$$

$\psi_n(\mathbf{X})$ represents the quark-antiquark bound state

Connection with quantum mechanics: $\psi_n(\mathbf{X}) \to \Psi(\mathbf{X},\mathbf{x}) = \Psi_{\mathbf{x}}(\mathbf{X})$
Ansatz: Promote $\Psi(\mathbf{X},\mathbf{x})$ to a field: $\Psi(\mathbf{X},\mathbf{x}) = \sum_n \phi_n(\mathbf{x})\psi_n(\mathbf{X})$,
where $\phi_n(\mathbf{x})$ is a function and $\psi_n(\mathbf{X})$ a field and $\hat{h}_{\mathbf{x}}\phi_n(\mathbf{x}) = E_n\phi_n(\mathbf{x})$

$$Z = \int \Pi D\psi_n^\dagger D\psi_n e^{i\int d^4X(\mathcal{L}+\psi_n^\dagger\eta_n+\eta_n^\dagger\psi_n)}$$

$$Z = \int D\Psi(X,x)^\dagger D\Psi(X,x) e^{i\int d^4X d^3\mathbf{x}(\mathcal{L}+\Psi^\dagger J(X,x)+J^\dagger(X,x)\Psi)}$$

$$\mathcal{L} = \Psi^\dagger\left(i\partial_0 + \frac{\boldsymbol{\nabla}_X^2}{2M} + \frac{\boldsymbol{\nabla}_x^2}{2\mu_r} - V(\mathbf{x}) + i\epsilon\right)\Psi$$

We have traded E_n for $\hat{h}_{\mathbf{x}} = -\frac{\boldsymbol{\nabla}_x^2}{2\mu_r} + V(\mathbf{x})$. Why?
To relate V($\mathbf{x}$) with some Green functions in the underlying theory and in some kinematical regime to compute it perturbatively.

Infinite number of states $\leftrightarrow$ Integro-diferential equation (Schrödinger equation)

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So...

We want to obtain an effective field theory, Potential Non-Relativistic QCD, which describes the heavy quarkonium dynamics and profits from the hierarchy $m \gg mv \gg mv^2$

$$\left.\begin{array}{l}\left(i\partial_0 - \frac{\mathbf{p}^2}{2m} - V_s^{(0)}(r)\right)\Phi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with other low} \\ \quad\text{energy degrees of freedom}\end{array}\right\} \text{potential NRQCD} \qquad E \sim mv^2$$

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Our aim is to provide a systematic method to deal with NR bound state systems. We will introduce a hierarchy of EFTs when sequentially integrating out each scale (only one scale in each step, strong simplification).

QCD

↓ Integrating out the hard scale (m)

NRQCD — Caswell, Lepage

↓ Integrating out the soft scale (mv)

pNRQCD — Soto, Pineda

NRQCD: the scale *m*

- Degrees of freedom
- Symmetries
- Cutoff

NRQCD has an ultraviolet cutoff Λ such that $m \gg \Lambda$ and larger than any other dynamical scale in the problem. $\Psi = \psi + \chi$

$$\mathcal{L}_{NRQCD} = \bar{\Psi} i \gamma^0 D_0 \Psi + \bar{\Psi} \left\{ \frac{\mathbf{D}^2}{2m} + c_F \, g \frac{\mathbf{\Sigma} \cdot \mathbf{B}}{2m} + c_D \, g \frac{\gamma^0 \left(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D}\right)}{8m^2} \right.$$
$$\left. + i c_S \, g \frac{\gamma^0 \mathbf{\Sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} + \frac{\mathbf{D}^4}{8m^3} \right\} \Psi$$
$$- \frac{1}{4} c_1 F_{\mu\nu} F^{\mu\nu} + \frac{c_2}{m^2} g F_{\mu\nu} D^2 g F^{\mu\nu} + \frac{c_3}{m^2} g^3 f_{ABC} F^A_{\mu\nu} F^B_{\mu\alpha} F^C_{\nu\alpha}$$

$$\delta \mathcal{L}_{NRQCD} = \frac{d_{ss}}{m_1 m_2} \psi_1^\dagger \psi_1 \chi_2^\dagger \chi_2 + \frac{d_{sv}}{m_1 m_2} \psi_1^\dagger \boldsymbol{\sigma} \psi_1 \chi_2^\dagger \boldsymbol{\sigma} \chi_2$$
$$+ \frac{d_{vs}}{m_1 m_2} \psi_1^\dagger \mathrm{T}^a \psi_1 \chi_2^\dagger \mathrm{T}^a \chi_2 + \frac{d_{vv}}{m_1 m_2} \psi_1^\dagger \mathrm{T}^a \boldsymbol{\sigma} \psi_1 \chi_2^\dagger \mathrm{T}^a \boldsymbol{\sigma} \chi_2 \, .$$

Lepage, Caswell, Thacker

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NRQED: the scale m

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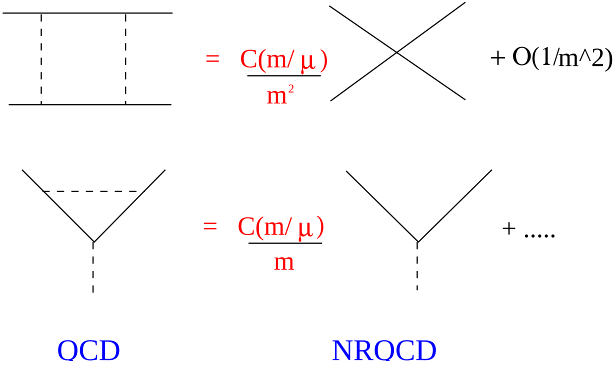
Matching QCD to NRQCD: the scale m

$c_i = 1 + O(\alpha_s)$, $d_1 = 1 + O(\alpha_s^2)$ (relevant α_s at low energies), $d's = O(\alpha_s)$.

$$c_i \sim 1 + \alpha_s \left(A \log \frac{m}{\mu} + B \right) \qquad d_i \sim \alpha_s \left(1 + \alpha_s \left(A \log \frac{m}{\mu} + B \right) \right)$$

One matches loops in QCD with only one scale (the mass) to tree level diagrams in NRQCD.



Manohar; Soto, Pineda

pNRQCD: the scale mv

The integration of the mv scale gives rise to potential terms. The Lagrangian is local in time but not in space.

- Degrees of freedom
- symmetries
- cutoff

pNRQCD has two ultraviolet cut-offs, ν_{us} and ν_p. ν_{us} fulfils the relation $\mathbf{p}^2/m \ll \nu_{us} \ll |\mathbf{p}|$ and is the cut-off of the energy of the quarks, and of the energy and the momentum of the gluons. ν_p fulfils $|\mathbf{p}| \ll \nu_p \ll m$ and is the cut-off of the relative momentum of the quark–antiquark system, $\mathbf{p}$.

Power counting/scales

Scales: m, p, $1/r$, $\Lambda_{mp} = \{\Lambda_{QCD}, mv^2, ...\}$

Dimensionless quantities:

$$\boxed{\frac{p}{m},\ \alpha_s,\ \frac{1}{m\,r},\ \boxed{\Lambda_{mp} r} \ll 1}$$

The multipole expansion can be used in the new EFT.

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$$L_{pNRQCD} = L'_{NRQCD} + \int\int d^3x_1 d^3x_2 \psi(x_1)\chi_c(x_2) V(x_1 - x_2)\psi^\dagger(x_1)\chi_c^\dagger(x_2)$$

L'_{NRQCD}, gluons multipole expanded (only ultrasoft gluons).

$$V_s^{(0)} \equiv -C_F \frac{\alpha_{V_s}}{r}.$$

$$\frac{V_s^{(1)}}{m} \equiv -\frac{C_F C_A D_s^{(1)}}{2mr^2}.$$

$$\frac{V_s^{(2)}}{m^2} = -\frac{C_F D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r}, \mathbf{p}^2\right\} + \frac{C_F D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2 + \frac{\pi C_F D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r})$$
$$+\frac{4\pi C_F D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_F D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_F D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),$$

where $S_{12}(\hat{\mathbf{r}}) \equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\,\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ and $\mathbf{S} = \boldsymbol{\sigma}_1/2 + \boldsymbol{\sigma}_2/2$.

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$$\begin{aligned}\frac{V_s^{(2)}}{m^2} = & -\frac{C_F D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r}, \mathbf{p}^2\right\} + \frac{C_F D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2 + \frac{\pi C_F D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r}) \\ & + \frac{4\pi C_F D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_F D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_F D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),\end{aligned}$$

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To go to the wave function description one has to project to the quark-antiquark sector.

$$\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi(\mathbf{x}_1,\mathbf{x}_2)\psi(x_1)\chi_c(x_2)|0\rangle$$

$$H\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi(\mathbf{x}_1,\mathbf{x}_2)\psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|0\rangle = \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 (\hat{h}\Psi(\mathbf{x}_1,\mathbf{x}_2))\psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|0\rangle$$

For QED (multipole expansion)

$$\begin{aligned} L_{pNRQED} &= \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi^\dagger(\mathbf{x}_1,\mathbf{x}_2)\left(iD_0 + \frac{\mathbf{D}^2_{\mathbf{x}_1}}{2m_1} + \frac{\mathbf{D}^2_{\mathbf{x}_2}}{2m_2} - V(\mathbf{x},\mathbf{p})\right)\Psi(\mathbf{x}_1,\mathbf{x}_2) \\ &= \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi^\dagger(\mathbf{x}_1,\mathbf{x}_2)\left(i\partial_0 + \frac{\boldsymbol{\nabla}^2_{\mathbf{x}}}{m} + \frac{\boldsymbol{\nabla}^2_{\mathbf{X}}}{4m}\right. \\ &\quad \left. -e\mathbf{x}\cdot\boldsymbol{\nabla}A_0(\mathbf{X}) - 2ie\frac{\mathbf{A}(\mathbf{X})\cdot\boldsymbol{\nabla}_{\mathbf{x}}}{m} - V(\mathbf{x},\mathbf{p})\right)\Psi(\mathbf{x}_1,\mathbf{x}_2) \end{aligned}$$

Field Redefinitions: $\Psi(\mathbf{x}_1,\mathbf{x}_2) = \phi(\mathbf{x}_1,\mathbf{x}_2)S(\mathbf{x},\mathbf{X})$

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New fields: Singlet S and US photons

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New fields: Singlet S and US gluons (and Octet (O) for QCD)

Gauge transformation:

$S(\mathbf{x},\mathbf{X},t) \to S(\mathbf{x},\mathbf{X},t) \quad O(\mathbf{x},\mathbf{X},t) \to g(\mathbf{X},t)O(\mathbf{x},\mathbf{X},t)g^{-1}(\mathbf{X},t)$

pNRQED Lagrangian at $O(r)$

$$\begin{aligned}\mathcal{L}_{pNRQED} &= S^\dagger\left(i\partial_0 - V_s^{(0)}(\mathbf{x})\right)S \\ &+ gV_A(\mathbf{x})S^\dagger\mathbf{x}\cdot\mathbf{E}\,S \\ &- S^\dagger\left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}(\mathbf{x})}{m^n}\right)S,\end{aligned}$$

$$V = V(c(\nu_s/m), d(\nu_s/m, \nu_p/m), r, \nu_s, \nu_{us}) \sim \sum_n c_n(\nu; m, r)\alpha_s^n(\nu)$$

Interpolating fields:

$$Q_2^\dagger(\mathbf{x}_2, t)\phi(\mathbf{x}_2, \mathbf{x}_1; t)Q_1(\mathbf{x}_1, t) = Z_s^{1/2}(\mathbf{x})S(\mathbf{X}, \mathbf{x}, t)$$

pNRQCD Lagrangian at $O(r)$

$$\mathcal{L}_{pNRQCD} = \mathrm{Tr}\left\{ S^\dagger \left(i\partial_0 - V_s^{(0)}(\mathbf{x}) \right) S + O^\dagger \left(iD_0 - V_o^{(0)}(\mathbf{x}) \right) O \right\}$$
$$+ gV_A(\mathbf{x})\mathrm{Tr}\left\{ O^\dagger \mathbf{x}\cdot\mathbf{E}\, S + S^\dagger \mathbf{x}\cdot\mathbf{E}\, O \right\} + g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{ O^\dagger \mathbf{x}\cdot\mathbf{E}\, O + O^\dagger O \mathbf{x}\cdot\mathbf{E} \right\}$$
$$-\mathrm{Tr}\left\{ S^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}(\mathbf{x})}{m^n} \right) S - O^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_o^{(n)}(\mathbf{x})}{m^n} \right) O \right\},$$

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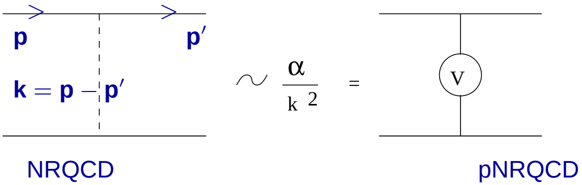
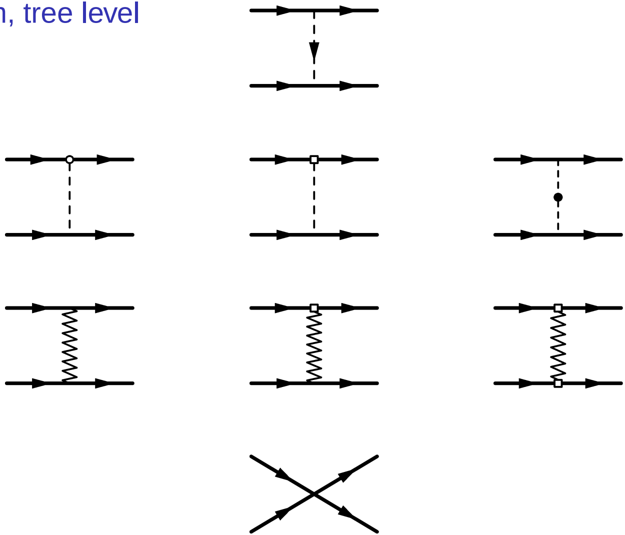
$$Q_2^\dagger(x_2)\phi(\mathbf{x}_2, \mathbf{X}; t)T^a\phi(\mathbf{X}, \mathbf{x}_1; t)Q_1(x_1) = Z_o^{1/2}(\mathbf{x})O^a(\mathbf{X}, \mathbf{x}, t)$$

Matching NRQCD to pNRQCD



Positronium, tree level



Muonic Hydrogen: electron vacuum polarization

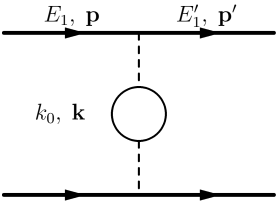


Figure: *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_p \alpha_V(k) \frac{1}{\mathbf{k}^2},$$

$$\alpha_{eff}(k) = \alpha \frac{1}{1 + \Pi(-\mathbf{k}^2)},$$

where

$$\Pi(k^2) = \alpha \Pi^{(1)}(k^2) + \alpha^2 \Pi^{(2)}(k^2) + \alpha^3 \Pi^{(3)}(k^2) + \ldots$$

$$\alpha_V(k) = \alpha_{eff}(k) + \sum_{\substack{n,m=0 \\ n+m=even>0}} Z_\mu^n Z_p^m \alpha_{eff}^{(n,m)}(k) = \alpha_{eff}(k) + \delta\alpha(k), \qquad \delta\alpha(k) = O(\alpha^4)$$

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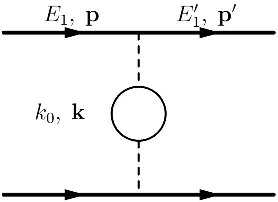

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Order $1/m^2$

$$\tilde{V}^{(b)} = \frac{\pi\alpha_{\rm eff}(k)}{2}\left[Z_p\frac{c_D^{(\mu)}}{m_\mu^2}+Z_\mu\frac{c_D^{(p)}}{m_p^2}\right],$$

$$\tilde{V}^{(c)} = -i2\pi\alpha_{\rm eff}(k)\frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left\{Z_p\frac{c_S^{(\mu)}\mathbf{s}_1}{m_\mu^2}+Z_\mu\frac{c_S^{(p)}\mathbf{s}_2}{m_p^2}\right\},$$

$$\tilde{V}^{(d)} = -Z_\mu Z_p 16\pi\alpha\left(\frac{d_2^{(\mu)}}{m_\mu^2}+\frac{d_2^{(\tau)}}{m_\tau^2}+\frac{d_{2,NR}}{m_p^2}\right),$$

$$\tilde{V}^{(e)} = -Z_\mu Z_p\frac{4\pi\alpha_{\rm eff}(k)}{m_\mu m_p}\left(\frac{\mathbf{p}^2}{\mathbf{k}^2}-\frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^4}\right),$$

$$\tilde{V}^{(f)} = -\frac{i4\pi\alpha_{\rm eff}(k)}{m_\mu m_p}\frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left(Z_p c_F^{(\mu)}\mathbf{s}_1+Z_\mu c_F^{(p)}\mathbf{s}_2\right),$$

$$\tilde{V}^{(g)} = \frac{4\pi\alpha_{\rm eff}(k)c_F^{(1)}c_F^{(2)}}{m_\mu m_p}\left(\mathbf{s}_1\cdot\mathbf{s}_2-\frac{\mathbf{s}_1\cdot\mathbf{k}\mathbf{s}_2\cdot\mathbf{k}}{\mathbf{k}^2}\right),$$

$$\tilde{V}^{(h)} = -\frac{1}{m_p^2}\left\{(c_{3,NR}^{pl_i}+3c_{4,NR}^{pl_i})-2c_{4,NR}^{pl_i}\mathbf{S}^2\right\}.$$

Order $1/m^2$ from energy-dependent terms

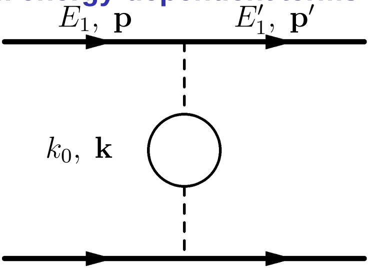
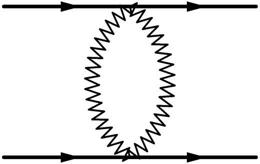
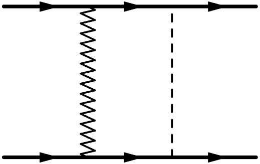


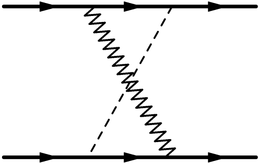
Figure: *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E'_1$*.*

$$\delta\tilde{V}_E = -\frac{Z_\mu Z_p e^2}{4m_\mu m_p}\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^2}\frac{\alpha}{\pi}m_e^2\int_4^\infty d(q^2)\frac{1}{(m_e^2 q^2+\mathbf{k}^2)^2}u(q^2)\,.$$

$$u(q^2) = \frac{1}{3}\sqrt{1-\frac{4}{q^2}}\left(1+\frac{2}{q^2}\right)\,.$$

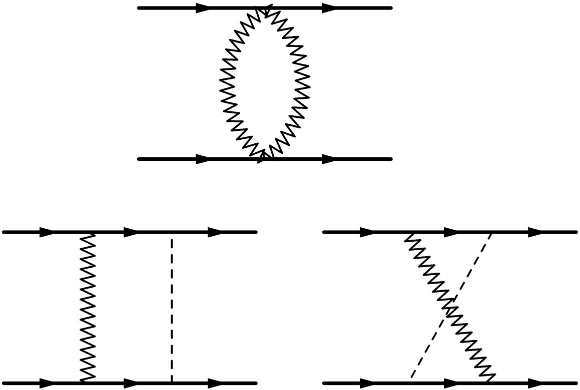






$$\tilde{V}^{(a)}_{1loop} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3}\log 2 + \frac{5}{3}\right),$$

$$\tilde{V}^{(b,c)}_{1loop} = \frac{4 Z_\mu^2 Z_p^2 \alpha^2}{3 m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} + 2\log 2 - 1\right).$$



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pNRQCD at last

Observable: Spectrum or decays

Corrections to the Green Function ($h_s^{(0)} = \mathbf{p}^2/m + V_s^{(0)}$)

$$G_s(E) = P_s \frac{1}{h_s^{(0)} - H_I - E} P_s = G_s^{(0)} + \delta G_s \qquad G_s^{(0)}(E) = \frac{1}{h_s^{(0)} - E}$$

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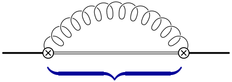
A) Ultrasoft loops (only at weak coupling: lamb shift-like): $\mathbf{x} \cdot \mathbf{E} \leftarrow$



$1/(E - V_o^{(0)} - \mathbf{p}^2/m)$

$$\delta G_s \sim \frac{1}{h_s^{(0)} - E} \int \frac{d^3\mathbf{k}}{(2\pi)^{D-1}} \mathbf{r} \frac{k}{k + h_o^{(0)} - E} \mathbf{r} \frac{1}{h_s^{(0)} - E}$$

$$\sim \frac{1}{h_s^{(0)} - E} \mathbf{r} (h_o^{(0)} - E)^3 \left\{ \frac{1}{\epsilon} + \gamma + \ln \frac{(h_o^{(0)} - E)^2}{\mu_{us}^2} + C \right\} \mathbf{r} \frac{1}{h_s^{(0)} - E}$$

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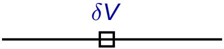
B) Quantum mechanics perturbation theory (both at weak and strong coupling)←

$$\delta G_s \sim \frac{1}{h_s^{(0)} - E} \delta V_s \frac{1}{h_s^{(0)} - E} + \frac{1}{h_s^{(0)} - E} \delta V_s \frac{1}{h_s^{(0)} - E} \delta V_s \frac{1}{h_s^{(0)} - E} + \cdots$$



Ultraviolet divergences are governed by the short-distance behavior of the potentials, i.e. by perturbation theory. Therefore, they can be computed and absorbed in the matching coefficients of the currents or in the potentials.

Summary (heavy quarkonium)

1) Matching QCD to NRQCD. Integrating out the hard scale, m
Relativistic Feynman diagrams $\leftarrow$
2) Matching NRQCD to pNRQCD. Integrating out the soft scale, mv
Potential = Wilson loops = HQET-like Feynman diagrams $\leftarrow$
3) Observable: Spectrum or decays
Corrections to the Green Function ($h_s^{(0)} = \mathbf{p}^2/m + V_s^{(0)}$)

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A) Ultrasoft loops (only at weak coupling: lamb shift-like): $\mathbf{x} \cdot \mathbf{E} \leftarrow$

B) Quantum mechanics perturbation theory (both at weak and strong coupling)$\leftarrow$

Summary (heavy quarkonium)

1) Matching QCD to NRQCD. Integrating out the hard scale, m
Relativistic Feynman diagrams ←
2) Matching NRQCD to pNRQCD. Integrating out the soft scale, mv
Potential = Wilson loops = HQET-like Feynman diagrams ←
3) Observable: Spectrum or decays
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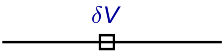

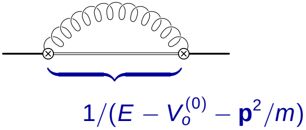

Summary (muonic hydrogen)

1) Matching HBET to NRQED. Integrating out the hard scale, $m_\mu \sim m_\pi$
HBET Feynman diagrams $\leftarrow$

2) Matching NRQED to pNRQED. Integrating out the soft scale, $m_\mu v$
Potential = Wilson loops = HQET-like Feynman diagrams $\leftarrow$

3) Observable: Spectrum or decays

Corrections to the Green Function ($h_s^{(0)} = \mathbf{p}^2/m + V_s^{(0)}$)

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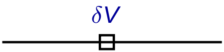

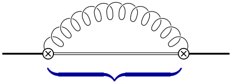

$$1/(E - V_s^{(0)} - \mathbf{p}^2/m)$$

Theoretical setup (muonic hydrogen Lamb shift)

We use an effective field theory, Potential Non-Relativistic QED, which describes the muonic hydrogen dynamics and profits from the hierarchy $m_\mu \gg m_\mu\alpha \gg m_\mu\alpha^2$

$$\left.\begin{array}{l}\left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r}\right)\psi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with ultrasoft photons}\end{array}\right\} \text{potential NRQED} \qquad E \sim mv^2$$

Scales:

$m_p \sim \Lambda_\chi$

$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p+m_\mu}$

$m_r\alpha \sim m_e$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r\alpha}{m_r} \sim \frac{m_r\alpha^2}{m_r\alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision $m_r\alpha^5$ (heavy quarkonium precision)

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Theoretical setup (muonic hydrogen Lamb shift: $\Delta E_L \equiv E(2P_{3/2}) - E(2S_{1/2})$)

$$L_{pNRQED} = \int d^3\mathbf{r} d^3\mathbf{R} dt S^\dagger(\mathbf{r},\mathbf{R},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{r}\cdot\mathbf{E}(\mathbf{R},t)\Big\} S(\mathbf{r},\mathbf{R},t) - \int d^3\mathbf{r} \frac{1}{4} F_{\mu\nu}F^{\mu\nu} ,$$

$$V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_p \alpha_V(k)\frac{1}{\mathbf{k}^2},$$

$$\alpha_{eff}(k) = \alpha\frac{1}{1+\Pi(-\mathbf{k}^2)},$$

where

$$\Pi(k^2) = \alpha\Pi^{(1)}(k^2) + \alpha^2\Pi^{(2)}(k^2) + \alpha^3\Pi^{(3)}(k^2) + \dots$$

$$\alpha_V(k) = \alpha_{eff}(k) + \sum_{\substack{n,m=0\\ n+m=even>0}} Z_\mu^n Z_p^m \alpha_{eff}^{(n,m)}(k) = \alpha_{eff}(k) + \delta\alpha(k), \qquad \delta\alpha(k) = O(\alpha^4)$$

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$$-V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{r}\cdot\mathbf{E}(\mathbf{R},t)\Big\}S(\mathbf{r},\mathbf{R},t) - \int d^3\mathbf{r}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,,$$

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Vacuum polarization effects: $\mathcal{O}(m_r\alpha^3)$

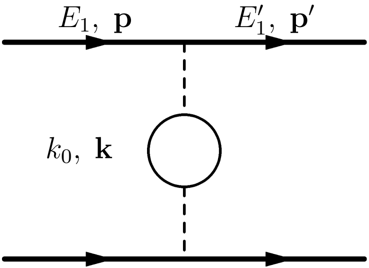
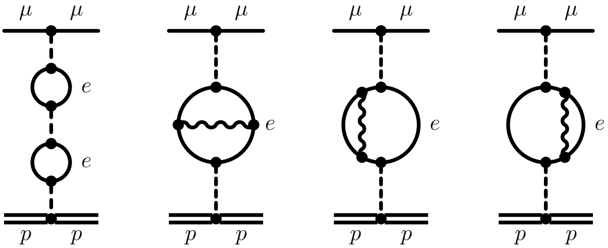


Figure: *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

1-loop static potential

$$E_{LO} = \langle n|\delta V|n\rangle = 205.0074\,\mathrm{meV} = \mathcal{O}(m_r\alpha^3)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



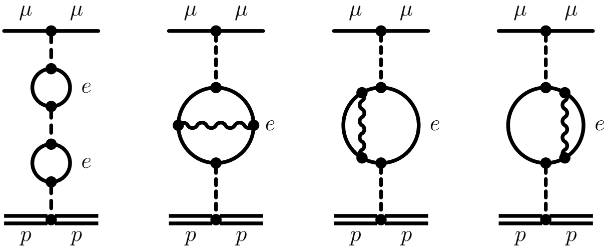
Pachuki/Borie
2-loop static potential is the same as two-loop vacuum polarization iterations
(*two loop vacuum polarization*)

$$\delta E = \langle n|\delta V|n\rangle = 1.5079\ \mathrm{meV} = \mathcal{O}(m_r\alpha^4)$$

Quantum mechanics perturbation theory (*iteration one-loop*)

$$\delta E \sim \langle n|\delta V \frac{1}{H_C - E_n}\delta V|n\rangle = 0.151\ \mathrm{meV} = \mathcal{O}(m_r\alpha^4)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



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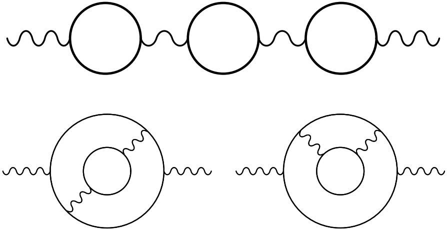
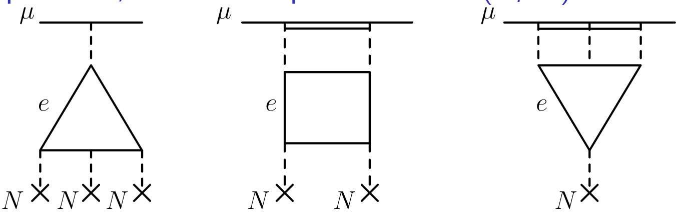


3-loop static potential (three loop vacuum polarization, Kinoshita-Nio)

$$0.00752\,\mathrm{meV} = \mathcal{O}(m_r\alpha^5)$$

Slightly corrected by Ivanov et al.

Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



(1:3) (2:2) (3:1)

Light-by-light (Wichmann-Kroll and Delbrück) contribution very small (Karshenboim et al.)

$$\Delta E \simeq -0.0009 \text{ mev} = \mathcal{O}(m_r\alpha^5)$$

Earlier work by Borie

Observation:
The limit $m_e \to 0$ known from QCD (Anzai et al. and Smirnov et al).
It should be possible to obtain the result with finite mass (albeit numerically) and check.

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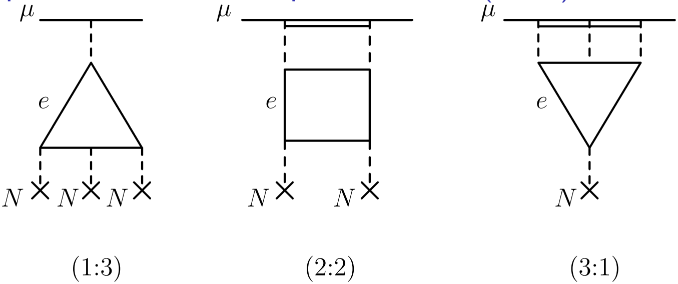

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$1/m$ potential

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu}F^{\mu\nu} ,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(1)}(r)}{m_\mu} \rightarrow \mathcal{O}(m_r\alpha^6)$$

relativistic corrections+vacuum polarization

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$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

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$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0169 (Pachucki and Veitia; Borie)

relativistic corrections+vacuum polarization

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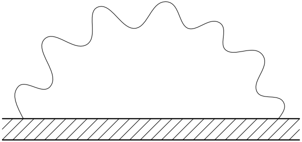
$\mathcal{O}(m\alpha^5)$ 0.018759 (Jentschura; Karshenboim, Ivanov, Korzinin)

Ultrasoft effects: $\mathcal{O}(m\alpha^5)$



$$\Delta E = -0.6677 \; meV$$

$$\mathcal{O}(m\alpha^5 \frac{m_\mu}{m_p}) : \qquad \Delta E = -0.045 \; meV$$

All (soft+ultrasoft):

$$\Delta E = -0.71896 \text{ meV}.$$

Start the overlap with hadronic effects.

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

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$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}\delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.}(m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \dots \frac{d_2}{m_p^2} F_{\mu\nu}D^2F^{\mu\nu} + \dots - e\frac{c_D}{m_p^2} N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E} N_p + \dots + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} -V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

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Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.}\delta^3(\mathbf{r}) \rightarrow \Delta E \sim \frac{1}{m_p^2} D_d^{had.}(m_r\alpha)^3$$

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c_3, d_2, c_D, ... matching coefficients of NRQED.

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Hadronic corrections

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Hadronic vacuum polarization effects

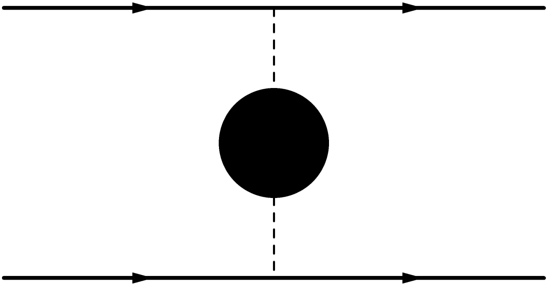
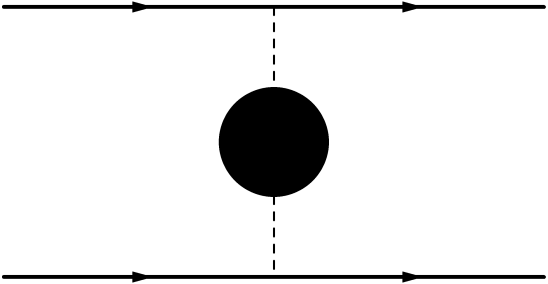


Figure: *Leading correction to the Coulomb potential due to the hadronic vacuum polarization.*

$d_2 \rightarrow$ hadronic vacuum polarization

$$\Delta E = 0.011 \text{ meV}$$

Hadronic vacuum polarization effects



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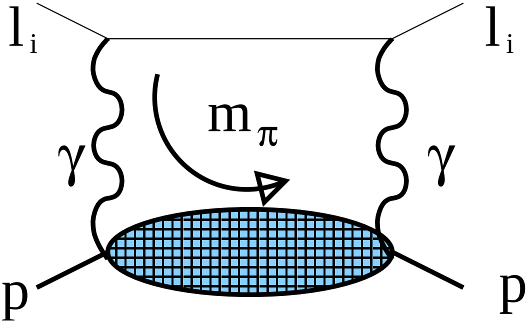

$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^\mu(x)J^\nu(0)|p,s\rangle\,,$$

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) S_1(\rho,q^2) + \frac{1}{m_p^2}\left(p^\mu - \frac{m_p\rho}{q^2}q^\mu\right)\left(p^\nu - \frac{m_p\rho}{q^2}q^\nu\right) S_2(\rho,q^2)$$

$$S_1 = ?? \qquad S_2 = ??$$

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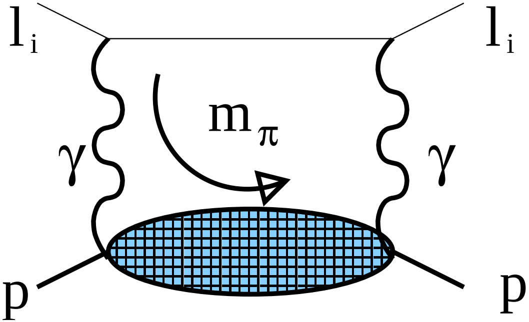

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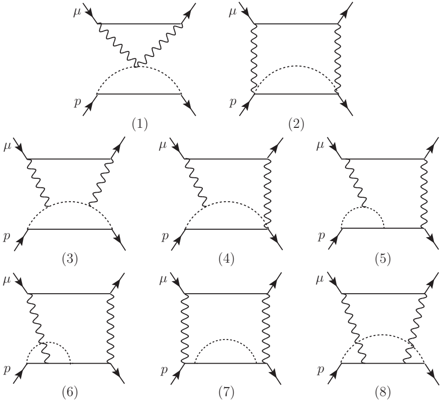
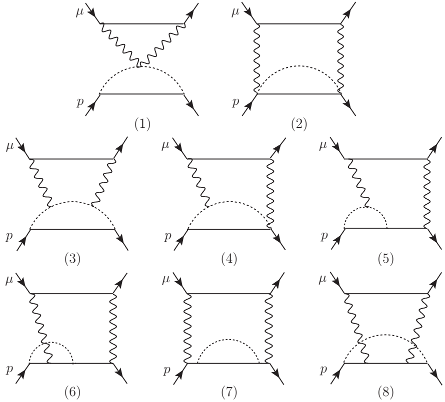


m_μ extra suppression+χPT (Model independent)

Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order!)

$$c_3^{\rm had} \sim \alpha^2 \frac{m_\mu}{m_\pi} + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) \qquad \delta E \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$$

Error: LO $\times \frac{m_\pi}{\Delta} \simeq$ LO $\times \frac{1}{2} \rightarrow c_3^{\rm had} = \frac{m_\mu}{m_\pi} 47.2(23.6)$



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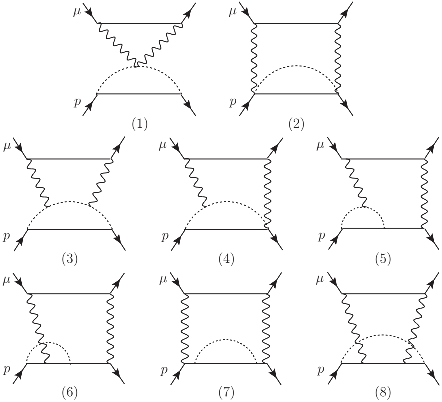

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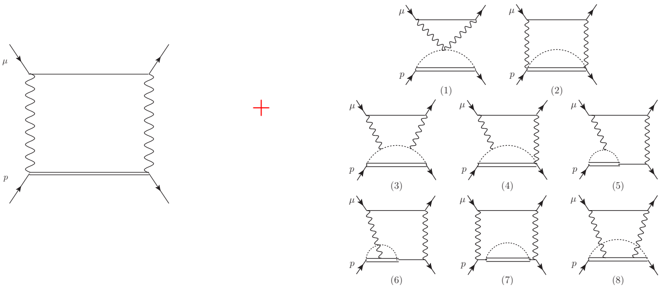
Large N_c. Including the Δ particle

Error:

$$\frac{m_\mu}{\Delta} \sim N_c \frac{m_\mu}{\Lambda_{QCD}} \rightarrow \frac{m_\mu}{\Lambda_{QCD}} \sim \frac{1}{3}$$



$$c_3^{\rm had} \sim \alpha^2 \frac{m_\mu}{m_\pi}\left[1 + \# \frac{m_\pi}{\Delta} + \cdots\right] + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) = \alpha^2 \frac{m_\mu}{m_\pi} \begin{cases} 47.2(23.6) & (\pi), \\ 56.7(20.6) & (\pi + \Delta), \end{cases}$$

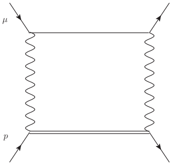
$$\Delta E_{\rm TPE} = 28.59(\pi) + 5.86(\pi\&\Delta) = 34.4(12.5)\mu{\rm eV} \;\; ({\rm Peset\&AP})\,.$$

(Model dependent: $\Delta E_{\rm TPE} = 33(2)\mu{\rm eV}$ (Birse-McGovern))

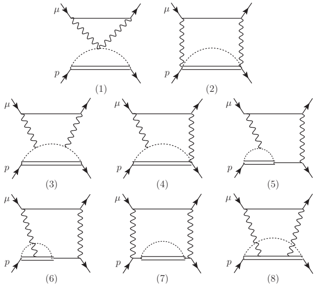
Large N_c. Including the Δ particle

Error:

$$\frac{m_\mu}{\Delta} \sim N_c \frac{m_\mu}{\Lambda_{QCD}} \rightarrow \frac{m_\mu}{\Lambda_{QCD}} \sim \frac{1}{3}$$



+



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$c_3^{\rm had}$: Two-Photon-Exchange contribution= Zemach+polarizability

Zemach:

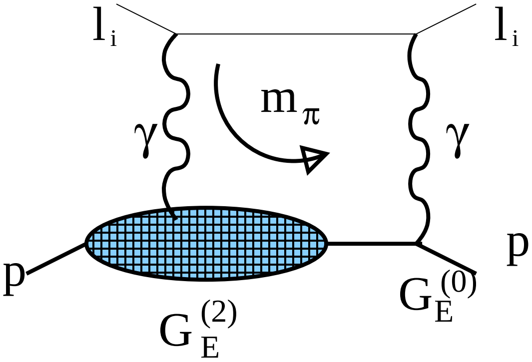

Figure: *Symbolic representation (plus permutations) of the Zemach $\langle r^3 \rangle$ correction.*

$$\Delta E_{\rm Zemach} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\rm fm^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\rm fm^3} = \frac{48}{\pi} \int \frac{d^3k}{4\pi} \frac{1}{\mathbf{k}^6} \left(G_E^2 - 1 + \frac{1}{3} \langle r^2 \rangle \mathbf{k}^2 \right) = \frac{96}{\pi} \int \frac{d^{D-1}k}{4\pi} \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

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$$\delta c_{3,Zemach}^{pl_i} = \frac{\pi}{3}\alpha^2 m_p^2 m_\mu \langle r_p^3 \rangle = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0}\right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta}\right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta}\right)^{2r} \right\},$$

where ($\Delta = M_\Delta - M_p \sim 300$ MeV)

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \qquad r \geq 0,$$

$$B_n \equiv \int_0^\infty dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln\left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1}\right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

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$$\frac{\langle r^3 \rangle_{(2)}}{\rm fm^3} = \frac{96}{\pi} \int d^{D-1}k \, \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\delta c^{pl_i}_{3,Zemach} = \frac{\pi}{3}\alpha^2 m_p^2 m_\mu \langle r_p^3 \rangle = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0}\right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} + \frac{2}{\pi} g^2_{\pi N\Delta} \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta}\right)^{2r} + g^2_{\pi N\Delta} \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta}\right)^{2r} \right\},$$

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Including Pions and Δ particles

	$\langle r^3 \rangle$	$\langle r^5 \rangle$	$\langle r^6 \rangle$	$\langle r^3 \rangle_{(2)}$
π	0.4980	1.619	5.203	0.9960
$\pi\&\Delta$	0.4071	0.6228	4.978	0.8142
Dipole	0.7706	1.775	3.325	2.023
Kelly	0.9838	3.209	7.440	2.526
Distler et al.	1.16(4)	8.0(1.2)(1.0)	29.8(7.6)(12.6)	2.85(8)

Table: *The first two rows give the prediction from the effective theory (Peset&AP). The third row corresponds to the standard dipole fit with* $\langle r^2 \rangle = 0.6581$ *fm*^{3}. *The fourth and fifth rows correspond to different parameterizations of experimental data. For completeness, we also quote* $\langle r^3 \rangle_{(2)} = 2.71$ *fm*^{3} *from Friar.*

μeV	DR	*Pachucki*	*Carlson et al*	HBET	*Peset&AP*(π)	$(\pi\&\Delta)$
ΔE_{Born}		23.2(1.0)	24.7(1.6)		10.1(5.1)	8.3(4.3)

Table: *Predictions for the Born contribution to the* $n = 2$ *Lamb shift. The first two entries correspond to dispersion relations. The last two entries are the predictions of HBET: The 3rd entry is the prediction of HBET at leading order (only pions) and the last entry is the prediction of HBET at leading and next-to-leading order (pions and Deltas).*

c_3 Polarizability effects

(μeV)	[1]	[2]	[3]	[4]	BχPT(π)	HBET(π)	(π&Δ)
$\Delta E_{\rm pol}$	12(2)	11.5	7.4(2.4)	15.3(5.6)	$8.2(^{+1.2}_{-2.5})$	18.5(9.3)	26.2(10.0)

Table: *Predictions for the polarizability contribution to the $n = 2$ Lamb shift. The first four entries use dispersion relations for the inelastic term and different modeling functions for the subtraction term. [1] Pachucki, [2] Martynenko, [3] Carlson&Vanderhaeghen, [4] Gorchtein et al.. The 5th entry is the prediction obtained using BχPT (Alarcon et al.). The last two entries are the predictions of HBET (Nevado&AP and Peset&AP).*

Polarizability=Inelastic+subtraction

$$c^{pl_i}_{3,\rm sub} = -e^4 M_p m_{l_i}\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^4}\frac{1}{k_E^4+4m_{l_i}^2k_{0,E}^2}(3k_{0,E}^2+\mathbf{k}^2)S_1(0,-k_E^2)$$

$$\begin{aligned}c^{pl_i}_{3,\rm inel} &= -e^4 M_p m_{l_i}\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^4}\frac{1}{k_E^4+4m_{l_i}^2k_{0,E}^2}\\ &\times\left\{(3k_{0,E}^2+\mathbf{k}^2)(S_1(ik_{0,E},-k_E^2)-S_1(0,-k_E^2))-\mathbf{k}^2S_2(ik_{0,E},-k_E^2)\right\}\end{aligned}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

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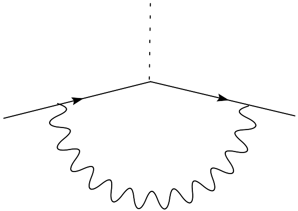
$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

Infrared divergent! $\rightarrow$ Wilson coefficient



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	$\mathcal{O}(m_r\alpha^3)$	$V_{\rm VP}^{(0)}$	205. 00737
	$\mathcal{O}(m_r\alpha^4)$	$V_{\rm VP}^{(0)}$	1. 50795
	$\mathcal{O}(m_r\alpha^4)$	$V_{\rm VP}^{(0)}$	0. 15090
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm VP}^{(0)}$	0. 00752
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm LbL}^{(0)}$	$-$0. 00089(2)
	$\mathcal{O}(m_r\alpha^4 \times \frac{m_\mu^2}{m_p^2})$	$V^{(2,1)} + V^{(3,0)}$	0. 05747
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm VP}^{(2,2)} + V^{(2,1)} \times V_{\rm VP}^{(0,2)}$	0. 01876
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm no-VP}^{(2,2)}$ + ultrasoft	$-$0. 71896
	$\mathcal{O}(m_r\alpha^6 \times \ln(\frac{m_\mu}{m_e}))$	$V^{(2,3)}; c_D^{(\mu)}$	$-$0. 00127
	$\mathcal{O}(m_r\alpha^6 \times \ln\alpha)$	$V_{\rm VP}^{(2,3)}; c_D^{(\mu)}$	$-$0. 00454
	$\mathcal{O}(m_r\alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}; c_D^{(p)}$	$-5.\ 1975\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^5 \times m_r^2 r_p^2)$	$V_{\rm VP}^{(2,2)} + V^{(2,1)} \times V_{\rm VP}^{(0,2)}; c_D^{(p)}$	$-0.\ 0282\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^6 \ln\alpha \times m_r^2 r_p^2)$	$V^{(2,3)}; c_D^{(p)}$	$-0.\ 0014\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2})$	$V_{\rm VP_{had}}^{(2)}; d_2^{\rm had}$	0. 0111(2)
	$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2}\frac{m_\mu}{m_\pi})$	$V^{(2)}; c_3^{\rm had}$	0. 0344(125)

$$\Delta E_L^{\rm our\ work} = \left[206.0243(30) - 5.2270(7)\frac{r_p^2}{\rm fm^2} + 0.0455(125)\right] \rm meV\,.$$

Using

$$\Delta E_L^{\rm exp} \equiv E(2P_{3/2}) - E(2S_{1/2}) = 202.3706(23)\,\rm meV \quad \text{Antognini et al.}$$

$$r_p = 0.8413(15)\,\rm fm.$$

At 6.8σ variance with respect the CODATA value.

CONCLUSIONS

Effective Field Theories can lead us to a better understanding of the dynamics of NR systems.
Model independent and systematic (Power counting).

Possible to obtain a rigorous connection between Quantum Field Theories and a NR Quantum-mechanical formulation of the NR systems.

Plenty of Observables: **Decay widhts ($\Gamma(n^3S_1 \to e^+e^-)$, $\Gamma(n^1S_0 \to \gamma\gamma)$), Bottomonium sum rules. Determination of m_b. t–$\bar{t}$ production near threshold. Determination of m_t. QED and atomic/hadronic physics, ...**

Proton radius: Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

The two-photon exchange energy shift (and the associated error) can only be computed in a model independent way with chiral perturbation theory. Overall number consistent with determinations from a combined use of dispersion relations and models, but individual contributions are quite different.

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$$\Delta E_L^{\text{our work}} = \left[206.0243(30) - 5.2270(7)\frac{r_p^2}{\text{fm}^2} + 0.0455(125)\right] \text{meV}\,.$$

Using

$$\Delta E_L^{\text{exp}} \equiv E(2P_{3/2}) - E(2S_{1/2}) = 202.3706(23)\,\text{meV} \quad \text{Antognini et al.}$$

$$r_p = 0.8413(15)\,\text{fm}.$$

At 6.8σ variance with respect the CODATA value.

Why going from NRQCD to pNRQCD?

Problem: Power counting of NRQCD in the perturbative situation.

Previous work: Labelle → Multipole expansion (QED)

Exploring the modes of the theory: Manohar, Luke; Grinstein, Rothstein; Savage, Luke

Different approach:

Soto, Pineda→How would we like the effective theory for Q–$\bar{Q}$ systems near threshold to be? We do not want to describe all the degrees of freedom included in NRQCD, but rather only those with US energy. Moreover, we want to get a closer connection with a Schrödinger-like formulation for these systems (also, eventually, in the non-perturbative regime → potential models). We name potential NRQCD this new effective theory.

Beneke, Smirnov→ threshold expansion. Rigorous diagrammatic study of the (perturbative) momentum regions: hard, soft, potential, ultrasoft.

hard: particles with $E \sim |\mathbf{p}| \sim m$ → NRQCD

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potential: particles with $E \sim mv^2$, $|\mathbf{p}| \sim mv$

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Implementation of pNRQCD through the threshold expansion (integrating out potential gluons and all soft particles): Beneke, Smirnov; Kniehl, Penin; Kniehl, Penin, Smirnov, Steinhauser

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Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \cdots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

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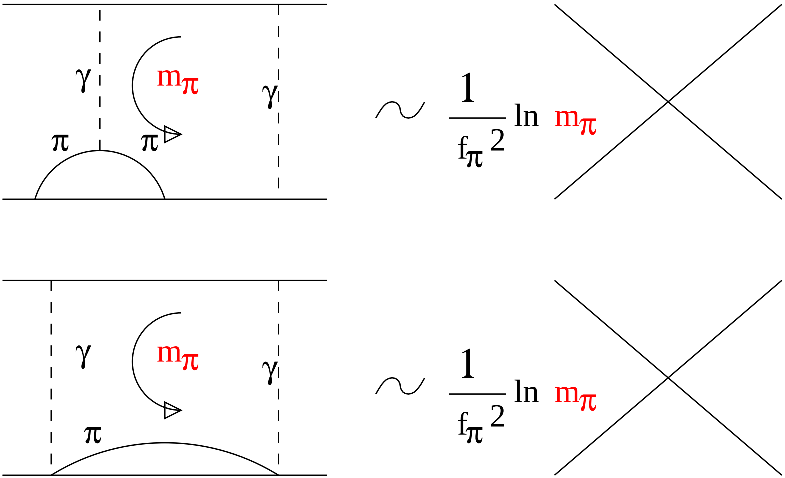
Leading chiral logs to the hyperfine splitting



$$\delta V = 2\frac{c_{4,NR}}{m_p^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r})\,.$$

c_4, Spin-dependent effects (Zemach): $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$

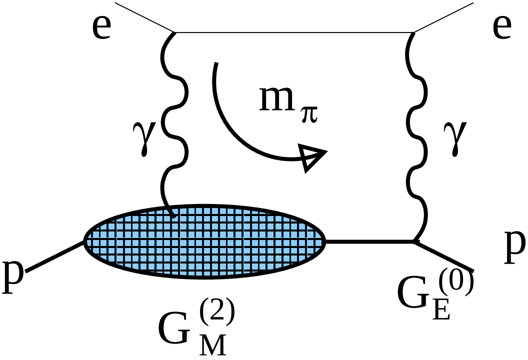


Figure: *Symbolic representation (plus permutations) of the Zemach correction.*

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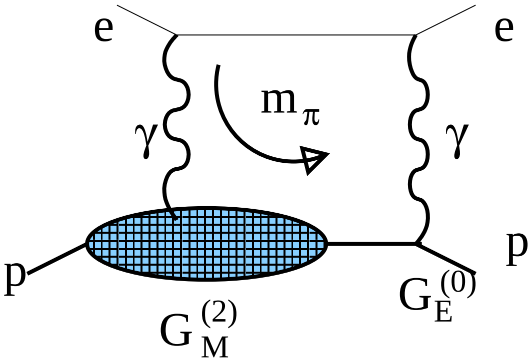

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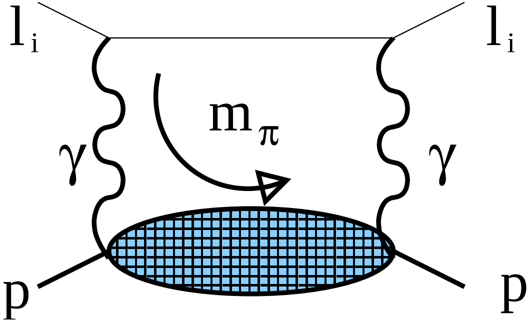

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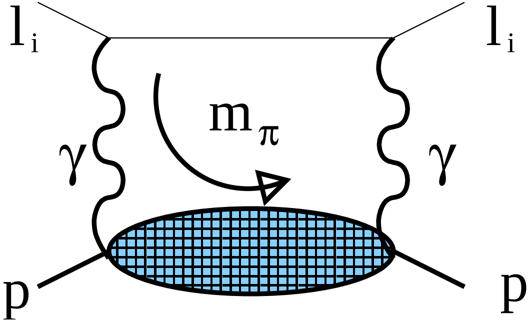

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$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^{\mu}(x)J^{\nu}(0)|p,s\rangle\,,$$

which has the following structure ($\rho = q\cdot p/m$):

$$\begin{aligned}
T^{\mu\nu} \quad = \quad & \left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)S_1(\rho,q^2) \\
& +\frac{1}{m_p^2}\left(p^{\mu}-\frac{m_p\rho}{q^2}q^{\mu}\right)\left(p^{\nu}-\frac{m_p\rho}{q^2}q^{\nu}\right)S_2(\rho,q^2) \\
& -\frac{i}{m_p}\,\epsilon^{\mu\nu\rho\sigma}q_{\rho}s_{\sigma}A_1(\rho,q^2) \\
& -\frac{i}{m_p^3}\,\epsilon^{\mu\nu\rho\sigma}q_{\rho}\big((m_p\rho)s_{\sigma}-(q\cdot s)p_{\sigma}\big)A_2(\rho,q^2)
\end{aligned}$$

Only logarithmically chiral enhanced but they can be determined from hydrogen hyperfine splitting.

$$\delta c_{4,NR}^{pl} \simeq \left(1-\frac{\mu_p^2}{4}\right)\alpha^2 \ln\frac{m_l^2}{\nu^2} + \frac{b_{1,F}^2}{18}\alpha^2\ln\frac{\Delta^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2\frac{2}{3}\left(\frac{2}{3}+\frac{7}{2\pi^2}\right)\pi^2 g_A^2 \ln\frac{m_\pi^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2\frac{8}{27}\left(\frac{5}{3}-\frac{7}{\pi^2}\right)\pi^2 g_{\pi N\Delta}^2 \ln\frac{\Delta^2}{\nu^2},$$

$$E_{\rm HF} = 4\frac{c_{4,NR}^{pl_i}}{m_p^2}\frac{1}{\pi}(\mu_{l_i p}\alpha)^3 \sim m_{l_i}\alpha^5\frac{m_{l_i}^2}{m_p^2}\times(\ln m_q, \ln\Delta, \ln m_{l_i}).$$

Hydrogen. By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\rm HF,logarithms}(m_\rho) = -0.031\ {\rm MHz}\,,$$

which accounts for approximately 2/3 of the difference between theory (pure QED) and experiment.

$$E_{\rm HF}(QED) - E_{\rm HF}(exp) = -0.046\ {\rm MHz}.$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c^{pl}_{4,NR} = -47.7\alpha^2$ and $c^{pl}_{4,R}(m_\rho) \simeq c^{p}_{4,R}(m_\rho) \simeq -16\alpha^2$.

Muonic hydrogen.

$$\Delta E_{HF} \simeq -0.153 meV\ (Pachucki : -0.145)$$

$$\Delta E = \frac{1}{4}(-0.15) meV = -0.0375 meV$$

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$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_n^2 = 6\frac{d}{dq^2}G_{n,E}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}c_D^{(n)}$$

$$c_D = 0 + 2F_2 + 8F_1' = 0 + 8m_n^2\left.\frac{dG_{n,E}(q^2)}{d\,q^2}\right|_{q^2=0}$$

Standard definition (corresponds to the experimental number):

$$r_n^2 = \frac{3}{4}\frac{1}{m_n^2}c_D$$

Neutron-lepton scattering length = REAL low energy constant

$$b_{nl} = \frac{1}{4m_n}\left(\alpha c_D - \frac{2}{\pi}c_{3,NR}^{nl}\right) \sim D_d^{(n)had}$$

It is not proportional to the radius

Definition of the neutron radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p} \right] u(p)$$

$$F_i(q^2) = 0 + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

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