Next-to-Next-to-leading ultrasoft running of the heavy quarkonium potentials and spectrum: Spin-independent case

Phys.Rev.D84:014012,2011

Antonio Pineda

Universitat Autònoma de Barcelona

QWG2011, Darmstadt, Germany October 4–7, 2011

Motivation

- t-$\bar{t}$ production near threshold.
- Non-relativistic sum rules. Determinations of the bottom quark mass.
- Attempts of a weak coupling description of higher excitations of heavy quarkonium:
  Brambilla-Sumino-Vairo
- Sumino-Recksiegel
  $1/m$ seemed to be important. Fine structure
- recent experimental determination of L=1 states
- Comparison with lattice potentials
- Also relevant for the muonic hydrogen lamb shift (proton radius)

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$$\mathcal{L}_{\rm us} = {\rm Tr}\Big\{ {\rm S}^\dagger \left(i\partial_0 - h_s(r)\right) {\rm S} + {\rm O}^\dagger \left(iD_0 - h_o(r)\right) {\rm O} \Big\}$$
$$+ gV_A(r){\rm Tr}\left\{ {\rm O}^\dagger \mathbf{r}\cdot\mathbf{E}\,{\rm S} + {\rm S}^\dagger \mathbf{r}\cdot\mathbf{E}\,{\rm O}\right\} + g\frac{V_B(r)}{2}{\rm Tr}\left\{ {\rm O}^\dagger \{\mathbf{r}\cdot\mathbf{E}, {\rm O}\}\right\}.$$

h_s can be splitted in the kinetic term and the potential:

$$h_s(\mathbf{r},\mathbf{p},\mathbf{S}_1,\mathbf{S}_2) = \frac{\mathbf{p}^2}{2m_r} + V_s(\mathbf{r},\mathbf{p},\mathbf{S}_1,\mathbf{S}_2) \equiv h_s^C + \delta h_s,$$
$$h_o(\mathbf{r},\mathbf{p},\mathbf{S}_1,\mathbf{S}_2) = \frac{\mathbf{p}^2}{2m_r} + V_o(\mathbf{r},\mathbf{p},\mathbf{S}_1,\mathbf{S}_2) \equiv h_o^C + \delta h_o,$$

where $h^C_{s/o} = \frac{\mathbf{p}^2}{2\,m_r} + V^C_{s/o}$, $m_r = m_1 m_2/(m_1+m_2)$ and $\mathbf{p} = -i\boldsymbol{\nabla}_{\mathbf{r}}$.

$$\begin{aligned} V_{s/o}(r) &= V^{(0)}_{s/o}(r) + \frac{V^{(1)}(r)}{m} + \frac{V^{(2)}}{m^2} + \cdots, \\ V^{(2)} &= V^{(2)}_{SD} + V^{(2)}_{SI}, \\ V^{(2)}_{SI} &= \frac{1}{2}\left\{\mathbf{p}^2, V^{(2)}_{\mathbf{p}^2}(r)\right\} + \frac{V^{(2)}_{\mathbf{L}^2}(r)}{r^2}\mathbf{L}^2 + V^{(2)}_r(r), \\ V^{(2)}_{SD} &= V^{(2)}_{LS}(r)\mathbf{L}\cdot\mathbf{S} + V^{(2)}_{S^2}(r)\mathbf{S}^2 + V^{(2)}_{\mathbf{S}_{12}}(r)\mathbf{S}_{12}(\hat{\mathbf{r}}), \end{aligned}$$

where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ and $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and $\mathbf{S}_{12}(\hat{\mathbf{r}}) \equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\,\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$.

$$\int dt e^{iEt} d^3R \langle vac|S(t,\mathbf{r},\mathbf{R})S^\dagger(0,\mathbf{r}',\mathbf{0})|vac\rangle$$
$$\sim \langle \mathbf{r}'|\frac{i}{E - h_s^B - \Sigma_B(E) + i\eta}|\mathbf{r}\rangle$$
$$\sim \phi_n(\mathbf{r})\phi_n(\mathbf{r}')\frac{i}{E - E_n^{pot} - \delta E_n^{us} + i\epsilon},$$

where n generically denotes the quantum number of the bound state: $n \to (n$ (principal quantum number), l (orbital angular momentum), s (total spin), j (total angular momentum)). E_n^{pot} and $\phi_n(\mathbf{r})$ are the eigenvalue and eigenfunction respectively of the equation

$$h_s\phi_n(\mathbf{r}) = E_n^{pot}\phi_n(\mathbf{r})$$

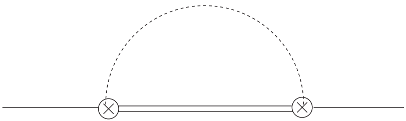
and, in general, will depend on the renormalization scheme the ultrasoft computation has been performed with.

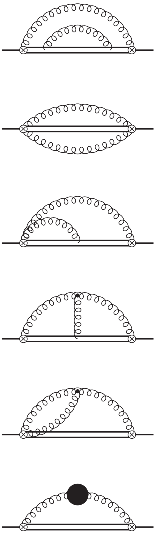
The self-energy $\Sigma_B(E)$ accounts for the effects due to the ultrasoft scale and can be expressed in a compact form at NLO in the multipole expansion (but exact to any order in α) through the chromoelectric correlator. It reads (in the Euclidean)

$$\Sigma_B(E) = V_A^2 \frac{T_F}{(D-1)N_c} \int_0^\infty dt \, \mathbf{r} e^{-t(h_o^B - E)} \mathbf{r} \langle vac | g\mathbf{E}_E^a(t) \phi_{\rm adj}^{ab}(t,0) g\mathbf{E}_E^b(0) | vac \rangle \, .$$

The pNRQCD one-loop computation yields

$$\Sigma_B(1-\mathrm{loop}) = -g^2 C_f V_A^2 (1+\epsilon) \frac{\Gamma(2+\epsilon)\Gamma(-3-2\epsilon)}{\pi^{2+\epsilon}} \mathbf{r} \, (h_o^B - E)^{3+2\epsilon} \mathbf{r} \, .$$





vNRQCD $\sim 10^4$ diagrams, pNRQCD $\sim$ 10 diagrams

The two-loop bare expression: Eidemuller-Jamin; Brambilla-Garcia-Soto-Vairo; Pineda-Stahlhofen

$$\Sigma_B(2-\mathrm{loop}) = g^4 C_f C_A V_A^2 \Gamma(-3-4\epsilon)\left[\mathcal{D}^{(1)}(\epsilon) - (1+2\epsilon)\mathcal{D}_1^{(1)}(\epsilon)\right]\mathbf{r}\,(h_o^B - E)^{3+4\epsilon}\mathbf{r}\,,$$

where

$$\mathcal{D}^{(1)}(\epsilon) = \frac{1}{(2\pi)^2}\frac{1}{4\pi^{2+2\epsilon}}\Gamma^2(1+\epsilon)g(\epsilon)\,,$$

$$\mathcal{D}_1^{(1)}(\epsilon) = \frac{1}{(2\pi)^2}\frac{1}{4\pi^{2+2\epsilon}}\Gamma^2(1+\epsilon)g_1(\epsilon)\,,$$

and

$$g(\epsilon) = \frac{2\epsilon^3+6\epsilon^2+8\epsilon+3}{\epsilon\,(2\epsilon^2+5\epsilon+3)} - \frac{2\epsilon\Gamma(-2\epsilon-2)\Gamma(-2\epsilon-1)}{(2\epsilon+3)\Gamma(-4\epsilon-3)}\,,$$

$$g_1(\epsilon) = \frac{6\epsilon^3+17\epsilon^2+18\epsilon+6}{\epsilon^2\,(2\epsilon^2+5\epsilon+3)} + \frac{4(\epsilon+1)n_f T_F}{\epsilon(2\epsilon+3)N_c} + \frac{2\left(\epsilon^2+\epsilon+1\right)\Gamma(-2\epsilon-2)\Gamma(-2\epsilon-1)}{\epsilon(2\epsilon+3)\Gamma(-4\epsilon-3)}\,.$$

From $\Sigma_B(E)$ it is possible to obtain δE_n^{us}.

$$
\begin{aligned}
\mathbf{r}(h_o - E)^3\mathbf{r} = \mathbf{r}^2(\Delta V)^3 - \frac{1}{2m_r^2}\left[\mathbf{p},\left[\mathbf{p},V_o^{(0)}\right]\right] + \frac{1}{2m_r^2}\left\{\mathbf{p}^2,\Delta V\right\} \\
+\frac{2}{m_r}\Delta V\left(r\frac{d}{dr}V_s^{(0)}\right) + \frac{1}{2m_r}(\Delta V)^2(3d-5) \\
+\frac{1}{2m_r}\left[4\Delta V\left(\left(r\frac{d}{dr}\Delta V\right)+\Delta V\right)+\left(\left(r\frac{d}{dr}\Delta V\right)+\Delta V\right)^2\right] \\
+\mathcal{O}((h_s - E)),
\end{aligned}
$$

where we have approximated $h_o - h_s = V_o^{(0)} - V_s^{(0)}$

Care with the D-dimensional dependence of V_s and $\Delta V = V_o - V_s$.

$$\delta V_s = \Bigg(\mathbf{r}^2(\Delta V)^3 - \frac{1}{2m_r^2}\left[\mathbf{p},\left[\mathbf{p},V_o^{(0)}\right]\right] + \frac{1}{2m_r^2}\left\{\mathbf{p}^2,\Delta V\right\} + \frac{2}{m_r}\Delta V\left(r\frac{d}{dr}V_s^{(0)}\right)$$
$$+\frac{\Delta V}{2m_r}\left[\Delta V(3d-5)+4\left(\left(r\frac{d}{dr}\Delta V\right)+\Delta V\right)+\left(\left(r\frac{d}{dr}\Delta V\right)+\Delta V\right)^2\right]\Bigg)$$
$$\times\left[\frac{1}{\epsilon}C_f V_A^2\left[\frac{\alpha(\nu)}{3\pi}+\frac{\alpha^2(\nu)}{36\pi^2}(C_A(\frac{47}{3}+2\pi^2)-\frac{10}{3}T_F n_f)\right]+\frac{1}{\epsilon^2}C_f V_A^2\frac{2}{3}\beta_0\frac{\alpha^2(\nu)}{(4\pi)^2}\right].$$

$$\nu\frac{d}{d\nu}V_{s,\overline{\rm MS}} = B_{V_s}\,,$$

where

$$B_{V_s} = C_f V_A^2\Bigg[\mathbf{r}^2(\Delta V)^3 + \frac{2}{m_r}\left(\Delta V\left(r\frac{d}{dr}V_s^{(0)}\right)+(\Delta V)^2\right) - \frac{1}{2m_r^2}\left[\mathbf{p},\left[\mathbf{p},V_o^{(0)}\right]\right]$$
$$+\frac{1}{2m_r^2}\left\{\mathbf{p}^2,\Delta V\right\}\Bigg]\times\left[-\frac{2\alpha(\nu)}{3\pi}+\frac{\alpha^2(\nu)}{9\pi^2}(C_A(-\frac{47}{3}-2\pi^2)+\frac{10}{3}T_F n_f)+\mathcal{O}(\alpha^3)\right]$$

$$\delta V_{s,RG}(r;\nu,\nu_{us}) = \left[\left(\mathbf{r}^2(\Delta V)^3 + \frac{2}{m_r}\left(\Delta V\left(r\frac{d}{dr}V_s^{(0)}\right) + (\Delta V)^2\right)\right)F(\nu;\nu_{us})\right.$$
$$\left. - \frac{1}{2m_r^2}\left[\mathbf{p},\left[\mathbf{p},V_o^{(0)}(r)F(\nu;\nu_{us})\right]\right] + \frac{1}{2m_r^2}\left\{\mathbf{p}^2,\Delta V(r)F(\nu;\nu_{us})\right\}\right],$$

and

$$F(\nu;\nu_{us}) = C_f V_A^2 \frac{2\pi}{\beta_0}\left\{\frac{2}{3\pi}\ln\frac{\alpha(\nu_{us})}{\alpha(\nu)}\right.$$
$$\left. -(\alpha(\nu_{us})-\alpha(\nu))\left(\frac{8}{3}\frac{\beta_1}{\beta_0}\frac{1}{(4\pi)^2} - \frac{1}{27\pi^2}\left(C_A\left(47+6\pi^2\right) - 10T_F n_f\right)\right)\right\}.$$

Static limit known before. LL: Pineda-Soto: NLL: Brambilla-Garcia-Soto-Vairo.
Beyond static limit. LL: Pineda
NLL confirmed by Hoang-Stahlhofen in vNRQCD (see Stahlhofen's talk)

Initial conditions

$$V_{s,\overline{\rm MS}}^{(0),RG}(r;\nu,\nu_{us}) = V_{s,\overline{\rm MS}}^{(0)}(r;\nu) + \delta V_{s,RG}^{(0)}(r;\nu,\nu_{us}) \equiv -C_f \frac{\alpha_{V_s}(r;\nu,\nu_{us})}{r}\,,$$

where

$$\delta V_{s,RG}^{(0)}(r;\nu,\nu_{us}) = \mathbf{r}^2(\Delta V)^3 F(\nu;\nu_{us})\,.$$

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$$V^{(0)}_{s,\overline{\rm MS}}(r;\nu) = -\frac{C_f\,\alpha(\nu)}{r}\left\{1+\sum_{n=1}^{3}\left(\frac{\alpha(\nu)}{4\pi}\right)^n a_n(r;\nu)\right\}$$

with coefficients

$$\begin{aligned}
a_1(r;\nu) &= a_1 + 2\beta_0 \ln\left(\nu e^{\gamma_E} r\right)\,,\\
a_2(r;\nu) &= a_2 + \frac{\pi^2}{3}\beta_0^2 + \left(4a_1\beta_0 + 2\beta_1\right)\ln\left(\nu e^{\gamma_E} r\right) + 4\beta_0^2\,\ln^2\left(\nu e^{\gamma_E} r\right)\,,\\
a_3(r;\nu) &= a_3 + a_1\beta_0^2\pi^2 + \frac{5\pi^2}{6}\beta_0\beta_1 + 16\zeta_3\beta_0^3\\
&+ \left(2\pi^2\beta_0^3 + 6a_2\beta_0 + 4a_1\beta_1 + 2\beta_2 + \frac{16}{3}C_A^3\pi^2\right)\ln\left(\nu e^{\gamma_E} r\right)\\
&+ \left(12a_1\beta_0^2 + 10\beta_0\beta_1\right)\ln^2\left(\nu e^{\gamma_E} r\right) + 8\beta_0^3\ln^3\left(\nu e^{\gamma_E} r\right)\,,
\end{aligned}$$

$\mathcal{O}(\alpha)$ Fischler; $\mathcal{O}(\alpha^2)$ Schroder; $\mathcal{O}(\alpha^3)$ logarithmic term Brambilla et al., Kniehl et al.; the light-flavour finite piece Smirnov et al.; and the pure gluonic finite piece Anzai et al., Smirnov et al..

For the $1/m$ potential the initial matching condition reads

$$V^{(1)}_{\overline{\rm MS}}(r;\nu) = \frac{C_f\alpha(e^{-\gamma_E}/r)}{2r^2}\left(\frac{C_f}{2}-C_A+\frac{\alpha(1/r)}{\pi}\left[-\frac{2}{3}(C_A^2+2C_AC_f)\ln(\nu^2r^2e^{2\gamma_E}) -\frac{89}{36}C_A^2+\frac{17}{18}C_AC_f+\frac{49}{36}C_AT_Fn_f-\frac{2}{9}C_fT_Fn_f\right]\right).$$

$\mathcal{O}(\alpha^3)$ log-dependent term Kniehl et al., Brambilla et al.; The log-independent $\mathcal{O}(\alpha^3)$ term from Kniehl et al. changed to our scheme.

$$V^{(1),RG}_{\overline{\rm MS}}(r;\nu,\nu_{us}) = V^{(1)}_{\overline{\rm MS}}(r;\nu)+\delta V^{(1)}_{RG}(r;\nu,\nu_{us}) \equiv -\frac{C_fC_AD^{(1)}(r;\nu,\nu_{us})}{2r^2},$$

where

$$\delta V^{(1)}_{RG}(r;\nu,\nu_{us}) = 4\left(\Delta V\left(r\frac{d}{dr}V_s^{(0)}\right)+(\Delta V)^2\right)F(\nu;\nu_{us}).$$

For the momentum-dependent $1/m^2$ potential the Wilson coefficient reads at one loop

$$V^{(2)}_{p^2,\overline{\rm MS}}(r;\nu) = \frac{C_f\alpha(e^{-\gamma_E}/r)}{4}\left(-4-\frac{\alpha(1/r)}{\pi}\left(\frac{31}{9}C_A-\frac{20}{9}T_Fn_f+\frac{8}{3}C_A\ln(\nu^2r^2e^{2\gamma_E})\right)\right)$$

$\mathcal{O}(\alpha^2)$ log-dependent term Kniehl et al., Brambilla et al.; the log-independent $\mathcal{O}(\alpha^2)$ term Kniehl et al..

$$V^{(2),RG}_{p^2,\overline{\rm MS}}(r;\nu,\nu_{us}) = V^{(2)}_{p^2,\overline{\rm MS}}(r;\nu)+\delta V^{(2)}_{p^2,RG}(r;\nu,\nu_{us}) \equiv -C_fD_1^{(2)}(r;\nu,\nu_{us})\,,$$

where

$$\delta V^{(2)}_{p^2,RG}(r;\nu,\nu_{us}) = 4\Delta V(r)F(\nu;\nu_{us})$$

V_r depends logarithmically on the mass of the heavy quark through the Wilson coefficients inherited from NRQCD. It is convenient to write it in terms of the potential in momentum space (otherwise ill-defined distributions appear)

$$\tilde{V}^{(2)}_{r,RG}(q;\nu_p,\nu,\nu_{us}) = \tilde{V}^{(2)}_{r,\overline{\rm MS}}(q;\nu_p,\nu) + \delta\tilde{V}^{(2)}_{r,RG}(q;\nu,\nu_{us})$$

$$\begin{aligned}\tilde{V}^{(2)}_{r,\overline{\rm MS}}(q;\nu_p,\nu) &= \pi C_f\Bigg[\alpha(q)(1+c_D(\nu)-2c_F^2(\nu)) \\ &+\frac{1}{\pi}(d_{vs}(\nu_p,\nu)+3d_{vv}(\nu_p,\nu)+\frac{1}{C_f}(d_{ss}(\nu_p,\nu)+3d_{vv}(\nu_p,\nu))) \\ &+\delta\tilde{V}_{soft}(\nu,q)\Bigg]\,,\end{aligned}$$

$$\delta\tilde{V}_{soft} = \frac{\alpha^2}{\pi}\left[\left(\frac{9}{4}+\frac{25}{6}\ln\frac{\nu^2}{q^2}\right)C_A + \left(\frac{1}{3}-\frac{7}{3}\ln\frac{\nu^2}{q^2}\right)C_f\right].$$

$$\delta\tilde{V}^{(2)}_{r,RG}(q;\nu,\nu_{us}) = -2\mathbf{q}^2\tilde{V}^{(0)}_o(q)F(\nu;\nu_{us})$$

$$V^{(2),RG}_{r,\overline{\rm MS}}(r;\nu_p,\nu_{us}) \equiv \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \tilde{V}^{(2)}_{r,RG}(q;\nu_p,q,\nu_{us})$$

$$= \delta^3(\mathbf{r}) \left(\tilde{V}^{(2)}_{r,RG}(\nu_p;\nu_p,\nu_p,\nu_{us}) - (\ln\nu_p) q \frac{d}{dq} (\tilde{V}^{(2)}_{r,RG}(q;\nu_p,q,\nu_{us}))|_{q=\nu_p} \right)$$

$$-\frac{1}{4\pi} \left({\rm reg}\frac{1}{r^3} \right) q \frac{d}{dq} (\tilde{V}^{(2)}_{r,RG}(q;\nu_p,q,\nu_{us}))|_{q=\nu_p} + \cdots .$$

Only the term proportional to ${\rm reg}\frac{1}{r^3}$ contributes to the $l \neq 0$ states mass. Higher order terms in the Taylor expansion are subleading.

$$q\frac{d}{dq}\left(\frac{\tilde{V}^{(2)}_{r,RG}(q;\nu_p,q,\nu_{us})}{\pi C_f} \right)|_{q=\nu_p} = -\frac{\alpha^2(\nu_p)}{\pi} \frac{16}{3} (\frac{C_A}{2} - C_f) \left[1 + \ln \frac{\alpha(\nu_p)}{\alpha(\nu_{us})} \right]$$

$$+\frac{\alpha^2(\nu_p)}{\pi} \left[-\frac{\beta_0}{2} + \frac{2}{3} T_F n_f (c_D(\nu_p) + c_1^{hl}(\nu_p)) \right.$$

$$\left. +(\beta_0 - \frac{13}{3} C_A) c_F^2(\nu_p) + (\frac{14}{3} C_f - \frac{2}{3} C_A) c_k^2 \right] .$$

$$
\begin{aligned}
V^{RG}_{s,\overline{\rm MS}}(r;\nu_p,\nu_{us}) &= V^{(0),RG}_{s,\overline{\rm MS}}(r;1/r,\nu_{us}) + \frac{V^{(1),RG}_{\overline{\rm MS}}(r;1/r,\nu_{us})}{m} \\
&+ \frac{1}{m^2}\left(\frac{1}{2}\left\{\mathbf{p}^2, V^{(2),RG}_{\mathbf{p}^2,\overline{\rm MS}}(r;1/r,\nu_{us})\right\} + V^{(2),RG}_{r,\overline{\rm MS}}(r;\nu_p,\nu_{us})\right)
\end{aligned}
$$

$$E_{nljs}|_{l\neq 0,s=0} = \left(E^{pot}_{\overline{\rm MS},nljs} + \delta E^{us}_{\overline{\rm MS},nl}\right)|_{l\neq 0,s=0}\,,$$

where $E^{pot}_{\overline{\rm MS},nljs}$ is the eigenvalue of the equation

$$\left(\frac{p^2}{2m_r} + V^{RG}_{s,\overline{\rm MS}}\right)\phi_{njls}(\mathbf{r}) = E^{pot}_{\overline{\rm MS},nljs}\phi_{nljs}(\mathbf{r})\,,$$

$$\delta E^{us}_{\overline{\rm MS},nl} = C_f V_A^2 \langle n,l|\mathbf{r}(h_o - E_{n,l})^3\left[-\frac{\alpha}{9\pi}\left(6\ln\left(\frac{h_o - E_{n,l}}{\nu}\right) + 6\ln 2 - 5\right)\right.$$
$$+\frac{\alpha^2}{108\pi^2}\left(18\beta_0\ln^2\left(\frac{h_o - E_{n,l}}{\nu}\right) - 6\left(N_c\left(13+4\pi^2\right)\right.\right.$$
$$\left. -2\beta_0(-5+3\ln 2)\right)\ln\left(\frac{h_o - E_{n,l}}{\nu}\right)$$
$$+2C_A\left(84 - 39\ln 2 + 4\pi^2(2-3\ln 2) - 72\zeta(3)\right)$$
$$\left.\left. +\beta_0\left(67 + 3\pi^2 - 60\ln 2 + 18\ln^2 2\right)\right)\right]\mathbf{r}|n,l\rangle\,.$$

Conclusions

We have computed the NLL ultrasoft running of the spin-independent singlet potentials up to $\mathcal{O}(1/m^2)$.

This sets the stage for the complete analytic and numerical computation of the heavy quarkonium spectrum with N^3LL accuracy for $l \neq 0$ and $s = 0$ states.

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