

THE MUONIC HYDROGEN LAMB SHIFT AND THE PROTON RADIUS

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Universitat Autònoma de Barcelona

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Precise measurements in atomic physics → Learning about hadron structure

Hyperfine splitting (hydrogen atom):

$$E_{HF}^{exp} = E(n = 1, s = 1) - E(n = 1, s = 0) \quad (s = \text{total spin})$$

Nature (1972)

$$\nu_{HF} = \frac{E_{HF}}{h} = 1420.4057517667(9) \text{ MHz} \quad (13 \text{ digits})$$

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Precise measurements in atomic physics → Learning about hadron structure Lamb shift (muonic hydrogen)

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PSI: R. Pohl et al., Nature vol. 466, p. 213 (2010)

$$E_{exp} = 206.2949(32) \text{ meV}$$

$$E_{th} = 209.9779(49) - 5.2262 \frac{r_p^2}{\text{fm}^2} + 0.0347 \frac{r_p^3}{\text{fm}^3} \text{ meV} = 205.984 \text{ meV}$$

using CODATA value $r_p = 0.8768(69) \text{ fm}$.

$$E_{exp} - E_{th} = 0.311 \text{ meV}$$

New proposed value: $r_p = 0.84184(67) \text{ fm}$. 5 standard deviations!!

$$E_{LO} = 205.0074 = \mathcal{O}(m_r \alpha^3)$$

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Theoretical setup

We use an effective field theory, **Potential Non-Relativistic QED**, which describes the muonic hydrogen dynamics and profits from the hierarchy

$$m_\mu \gg m_\mu \alpha \gg m_\mu \alpha^2$$

$$\left. \begin{aligned} & \left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r} \right) \psi(\mathbf{r}) = 0 \\ & + \text{corrections to the potential} \\ & + \text{interaction with ultrasoft photons} \end{aligned} \right\} \text{potential NRQED} \quad E \sim mv^2$$

Scales:

$$m_p \sim \Lambda_\chi$$

$$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p + m_\mu}$$

$$m_r \alpha \sim m_e$$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r \alpha}{m_r} \sim \frac{m_r \alpha^2}{m_r \alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision $m_r \alpha^5$ (heavy quarkonium precision)

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$$L_{pNRQED} = \int d^3r d^3R dt S^\dagger(\mathbf{r}, \mathbf{R}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{r}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{r} \cdot \mathbf{E}(\mathbf{R}, t) \right\} S(\mathbf{r}, \mathbf{R}, t) - \int d^3r \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

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$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_\rho \alpha_V(k) \frac{1}{\mathbf{k}^2},$$

$$\alpha_{\text{eff}}(k) = \alpha \frac{1}{1 + \Pi(-\mathbf{k}^2)},$$

where

$$\Pi(k^2) = \alpha \Pi^{(1)}(k^2) + \alpha^2 \Pi^{(2)}(k^2) + \alpha^3 \Pi^{(3)}(k^2) + \dots$$

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Vacuum polarization effects: $\mathcal{O}(m_r\alpha^3)$

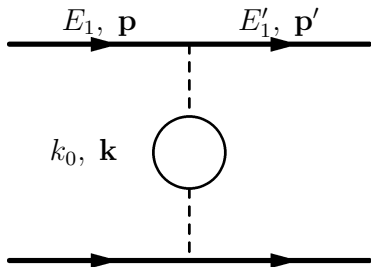
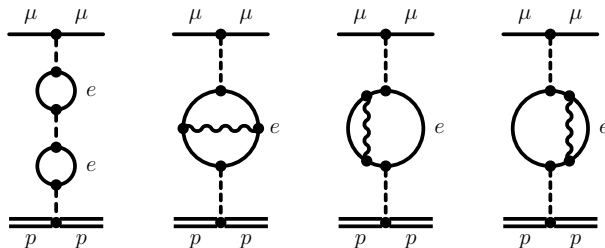


Figure: Leading correction to the Coulomb potential due to the electron vacuum polarization. $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ and $k_0 = E_1 - E'_1$.

1-loop static potential

$$E_{LO} = 205.0074 = \mathcal{O}(m_r\alpha^3)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4, m_r\alpha^5)$

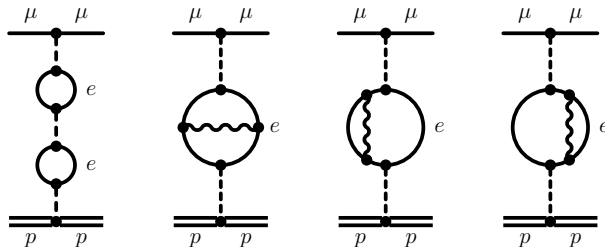


Pachuki/Borie

2-loop static potential is the same as two-loop vacuum polarization iterations
 1.5079 (*two loop vacuum polarization*)+ 0.151 (*iteration one-loop*)

3-loop static potential (three loop vacuum polarization, Kinoshita-Nio, 0.0076)

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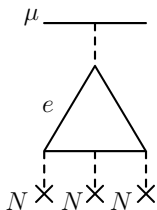


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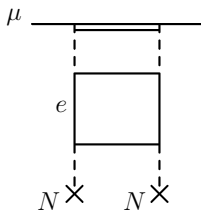
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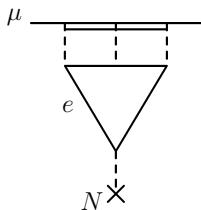
Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



(1:3)



(2:2)



(3:1)

Light-by-light (Wichmann-Kroll and Delbrück) contribution very small

$$\Delta E \simeq -0.0009 \text{ meV (Karshenboim *et al.*)}$$

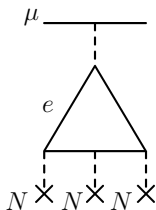
Earlier work by Borie

Observation:

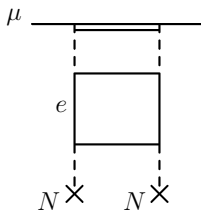
The limit $m_e \rightarrow 0$ known from QCD (Anzai *et al.* and Smirnov *et al.*)

It should be possible to obtain the result with finite mass (albeit numerically) and check.

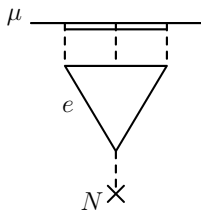
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1/m potential

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$\mathcal{O}(m\alpha^4)$ 0.0575 (purely relativistic)

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Matching NRQED to pNRQED. Getting the potential

relativistic corrections+vacuum polarization

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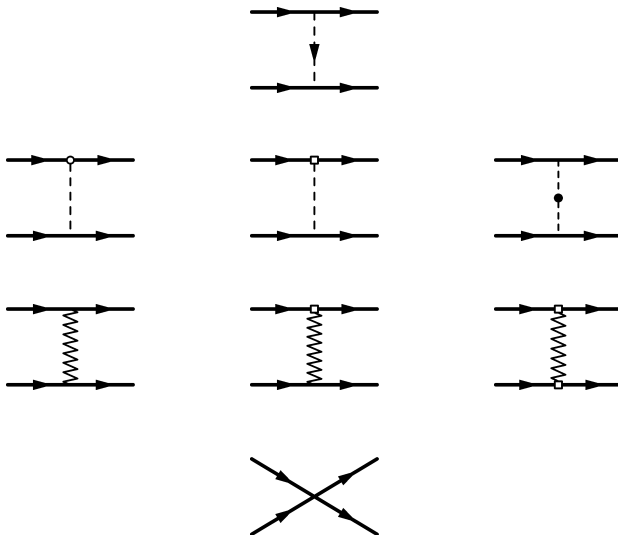
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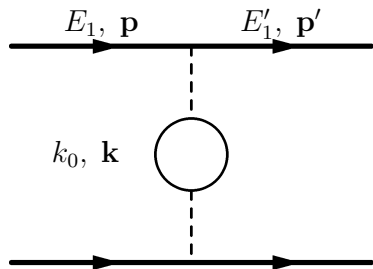


Figure: *Leading correction to the Coulomb potential due to the electron vacuum polarization. $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ and $k_0 = E_1 - E'_1$.*

Order $1/m^2$

$$\tilde{V}^{(b)} = \frac{\pi\alpha_{\text{eff}}(k)}{2} \left[Z_p \frac{c_D^{(\mu)}}{m_\mu^2} + Z_\mu \frac{c_D^{(p)}}{m_p^2} \right],$$

$$\tilde{V}^{(c)} = -i2\pi\alpha_{\text{eff}}(k) \frac{(\mathbf{p} \times \mathbf{k})}{\mathbf{k}^2} \cdot \left\{ Z_p \frac{c_S^{(\mu)} \mathbf{s}_1}{m_\mu^2} + Z_\mu \frac{c_S^{(p)} \mathbf{s}_2}{m_p^2} \right\},$$

$$\tilde{V}^{(d)} = -Z_\mu Z_p 16\pi\alpha \left(\frac{d_2^{(\mu)}}{m_\mu^2} + \frac{d_2^{(\tau)}}{m_\tau^2} + \frac{d_{2,NR}}{m_p^2} \right),$$

$$\tilde{V}^{(e)} = -Z_\mu Z_p \frac{4\pi\alpha_{\text{eff}}(k)}{m_\mu m_p} \left(\frac{\mathbf{p}^2}{\mathbf{k}^2} - \frac{(\mathbf{p} \cdot \mathbf{k})^2}{\mathbf{k}^4} \right),$$

$$\tilde{V}^{(f)} = -\frac{i4\pi\alpha_{\text{eff}}(k)}{m_\mu m_p} \frac{(\mathbf{p} \times \mathbf{k})}{\mathbf{k}^2} \cdot (Z_p c_F^{(\mu)} \mathbf{s}_1 + Z_\mu c_F^{(p)} \mathbf{s}_2),$$

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$$\tilde{V}^{(h)} = -\frac{1}{m_p^2} \left\{ (c_{3,NR}^{pl_j} + 3c_{4,NR}^{pl_j}) - 2c_{4,NR}^{pl_j} \mathbf{S}^2 \right\}.$$

Order $1/m^2$ from energy-dependent terms

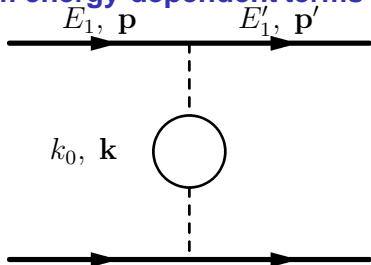


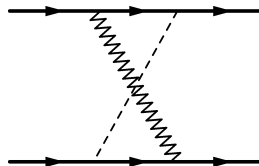
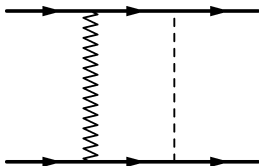
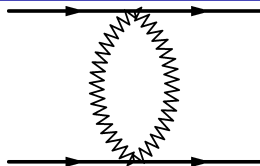
Figure: Leading correction to the Coulomb potential due to the electron vacuum polarization. $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ and $k_0 = E_1 - E_1'$.

$$\delta \tilde{V}_E = -\frac{Z_\mu Z_p e^2}{4m_\mu m_p} \frac{(\mathbf{p}^2 - \mathbf{p}'^2)^2}{\mathbf{k}^2} \frac{\alpha}{\pi} m_e^2 \int_4^\infty d(q^2) \frac{1}{(m_e^2 q^2 + \mathbf{k}^2)^2} u(q^2).$$

$$u(q^2) = \frac{1}{3} \sqrt{1 - \frac{4}{q^2}} \left(1 + \frac{2}{q^2}\right).$$

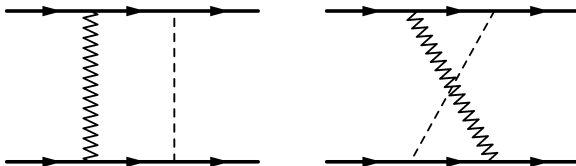
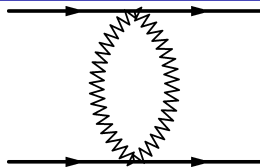
$$\delta V_E + V^{(e)} \Big|_{1\text{loop}} = -\frac{Z_\mu Z_p \alpha^2}{\pi} \frac{m_e^2}{m_\mu m_p} \int_4^\infty d(q^2) \frac{u(q^2)}{(m_e q)^2} \left\{ \frac{1}{2} \left\{ \mathbf{p}^2, \frac{e^{-m_e q r}}{r} \left(1 + \frac{m_e q r}{2} \right) \right\} - \frac{e^{-m_e q r}}{2r^3} (1 + m_e q r) \mathbf{L}^2 + \frac{(m_e q)^2}{4r} e^{-m_e q r} \left(1 + \frac{m_e q r}{2} \right) - 2\pi \delta(\mathbf{r}) \right\},$$

where \mathbf{L} is the angular momentum. It agrees with the corresponding expression by Pachucki.



$$\tilde{V}_{1loop}^{(a)} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p} \left(\log \frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3} \log 2 + \frac{5}{3} \right),$$

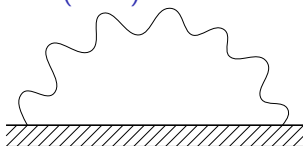
$$\tilde{V}_{1loop}^{(b,c)} = \frac{4Z_\mu^2 Z_p^2 \alpha^2}{3m_\mu m_p} \left(\log \frac{\mathbf{k}^2}{\mu^2} + 2 \log 2 - 1 \right).$$



$$\tilde{V}_{1loop}^{(a)} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p} \left(\log \frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3} \log 2 + \frac{5}{3} \right),$$

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Ultrasoft effects: $\mathcal{O}(m\alpha^5)$



Pachucki

$$\Delta E = -0.6677 \text{ meV}$$

$$\mathcal{O}(m\alpha^5 \frac{m_\mu}{m_p}) : \quad \Delta E = -0.045 \text{ meV}$$

Start the overlap with hadronic effects.

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.} \delta^3(\mathbf{r}) \rightarrow \Delta E \sim \frac{1}{m_p^2} D_d^{had.} (m_r \alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D$$

c_3, d_2, c_D, \dots matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \dots - e \frac{c_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p + \dots + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

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Hadronic corrections

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Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

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HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_I + \mathcal{L}_\pi + \mathcal{L}_{I\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)I} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)I\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \dots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\text{Tr}[D_\mu U D^\mu U] + \dots \quad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger (iv^\mu \nabla_\mu + g_A u_\mu S^\mu) N + \dots + (\Delta) + \dots - e\frac{C_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \quad \nabla_\mu = \partial_\mu + \Gamma_\mu \quad u_\mu = iu^\dagger (\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2} \left\{ u^\dagger (\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger \right\}$$

$$\mathcal{L}_{N,I} = \frac{1}{m_p^2} \sum_i c_{3,R}^{pli} \bar{N}_p \gamma^0 N_p \bar{l}_i \gamma^0 l_i + \frac{1}{m_p^2} \sum_i c_{4,R}^{pli} \bar{N}_p \gamma^j N_p \bar{l}_i \gamma_j l_i$$

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$$\delta\mathcal{L} = \dots - \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \dots - e\frac{C_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p + \dots + \frac{C_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

HBET (m_π)

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$$D_\mu = \partial_\mu + ieQA_\mu \quad \nabla_\mu = \partial_\mu + \Gamma_\mu \quad u_\mu = iu^\dagger (\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2} \left\{ u^\dagger (\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger \right\}$$

$$\mathcal{L}_{N,I} = \frac{1}{m_p^2} \sum_i c_{3,R}^{pl_i} \bar{N}_p \gamma^0 N_p \bar{l}_i \gamma^0 l_i + \frac{1}{m_p^2} \sum_i c_{4,R}^{pl_i} \bar{N}_p \gamma^j N_p \bar{l}_i \gamma_j l_i$$

$$\delta\mathcal{L} = \dots - \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \dots - e\frac{C_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p + \dots + \frac{C_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

Hadronic vacuum polarization effects

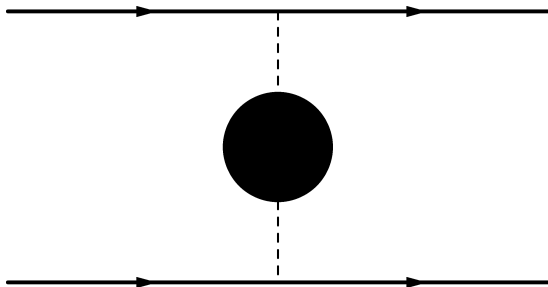


Figure: *Leading correction to the Coulomb potential due to the hadronic vacuum polarization.*

$d_2 \rightarrow$ hadronic vacuum polarization

$$\Delta E = 0.011 \text{ meV}$$

Hadronic vacuum polarization effects

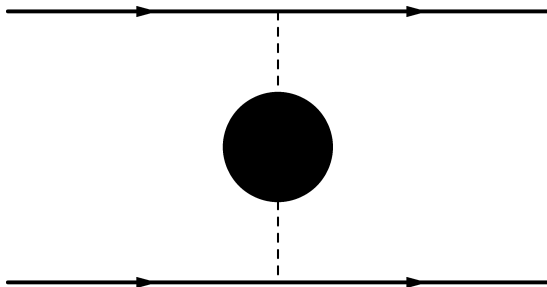


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c_3 or Zemach (r^3) effects: $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$

Power-like chiral enhanced ($\rightarrow \chi$ PT can predict the leading order)
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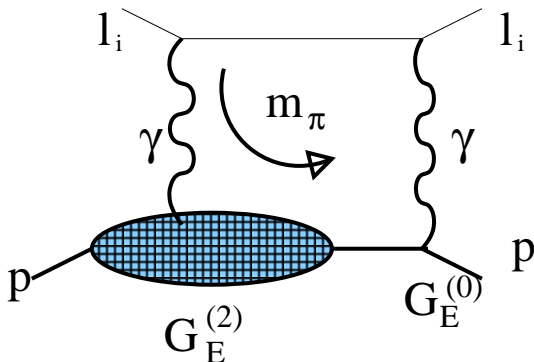


Figure: Symbolic representation (plus permutations) of the Zemach $\langle r^3 \rangle$ correction.

$$\Delta E = 0.010 \frac{\langle r_p^3 \rangle}{\text{fm}^3}$$

$$\frac{\langle r_p^3 \rangle}{\text{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\begin{aligned} \delta C_{3,Zemach}^{pl_i} &= \frac{\pi}{3} \alpha^2 m_p^2 m_\mu \langle r_p^3 \rangle = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0} \right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} \right. \\ &\quad \left. + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta} \right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta} \right)^{2r} \right\}, \end{aligned}$$

where $(\Delta = M_\Delta - M_p \sim 300 \text{ MeV})$

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \quad r \geq 0,$$

$$B_n \equiv \int_0^{\infty} dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln \left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1} \right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

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Not the reason for the discrepancy.

$\langle r_p^3 \rangle \sim 35$ De Rujula, not consistent neither with experiment nor chiral symmetry.

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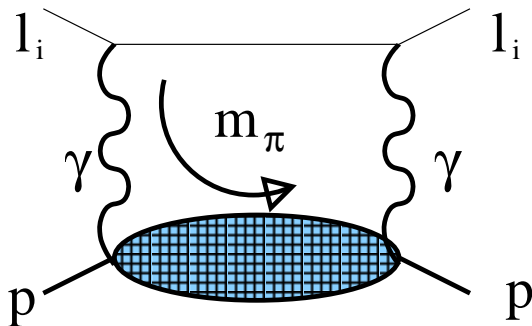
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m_μ extra suppression



$$\Delta E|_{Fitexp} = 0.012(\text{Pachucki})/0.015(\text{Borie}) \text{ meV}$$

$$\Delta E|_{\chi PT}(\text{pions}) = 0.018(\text{Nevado} - \text{Pineda}) \text{ meV}$$

$$c_{3,NR}^{pl_i} = -e^4 m_p m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \\ \times \left\{ (3k_{0,E}^2 + \mathbf{k}^2) S_1(ik_{0,E}, -k_E^2) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\}$$

$$T^{\mu\nu} = i \int d^4 x e^{iq \cdot x} \langle p, s | T J^\mu(x) J^\nu(0) | p, s \rangle,$$

which has the following structure ($\rho = q \cdot p/m$):

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) \\ + \frac{1}{m_p^2} \left(p^\mu - \frac{m_{p\rho}}{q^2} q^\mu \right) \left(p^\nu - \frac{m_{p\rho}}{q^2} q^\nu \right) S_2(\rho, q^2) \\ - \frac{i}{m_p} \epsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\rho, q^2) \\ - \frac{i}{m_p^3} \epsilon^{\mu\nu\rho\sigma} q_\rho \left((m_{p\rho}) s_\sigma - (q \cdot s) p_\sigma \right) A_2(\rho, q^2)$$

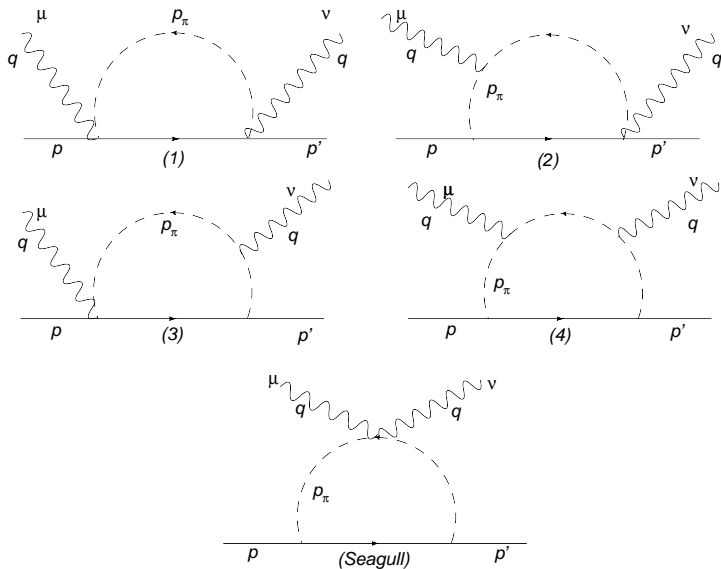


Figure: Diagrams contributing to T^{ij} . Crossed diagrams are not explicitly shown but calculated.

$$\begin{aligned}
c_{3,NR}^{pl_i} &= -e^4 m_p^2 \frac{m_{l_i}}{m_\pi} \left(\frac{g_A}{f_\pi} \right)^2 \int \frac{d^{D-1} k_E}{(2\pi)^{D-1}} \frac{1}{(1 + \mathbf{k}^2)^4} \\
&\times \int_0^\infty \frac{dw}{\pi} w^{D-5} \frac{1}{w^2 + 4 \frac{m_{l_i}^2}{m_\pi^2} \frac{1}{(1 + \mathbf{k}^2)^2}} \\
&\times \left\{ (2 + (1 + \mathbf{k}^2)^2) A_E(w^2, \mathbf{k}^2) + (1 + \mathbf{k}^2)^2 \mathbf{k}^2 w^2 B_E(w^2, \mathbf{k}^2) \right\}
\end{aligned}$$

where (for $D = 4$)

$$A_E = -\frac{1}{4\pi} \left[-\frac{3}{2} + \sqrt{1 + w^2} + \int_0^1 dx \frac{1 - x}{\sqrt{1 + x^2 w^2 + x(1 - x) w^2 \mathbf{k}^2}} \right],$$

$$\begin{aligned}
B_E &= \frac{1}{8\pi} \left[\int_0^1 dx \frac{1 - 2x}{\sqrt{1 + x^2 w^2 + x(1 - x) w^2 \mathbf{k}^2}} \right. \\
&\quad \left. - \frac{1}{2} \int_0^1 dx \frac{(1 - x)(1 - 2x)^2}{(1 + x^2 w^2 + x(1 - x) w^2 \mathbf{k}^2)^{\frac{3}{2}}} \right].
\end{aligned}$$

Definition of the proton radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

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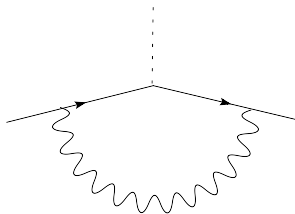
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$$r_p^2(\nu) = 6 \frac{d}{dq^2} G_{E,p}(q^2) |_{q^2=0}$$

Infrared divergent! → Wilson coefficient



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Neutron-lepton scattering length = REAL low energy constant

$$b_{nl} = \frac{1}{4m_n} \left(\alpha c_D - \frac{2}{\pi} c_{3,NR}^{nl} \right) \sim D_d^{(n)had}$$

It is **not** proportional to the radius

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Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

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$$D_s^{had.} = 2c_4$$

c_4 , matching coefficient of NRQED.

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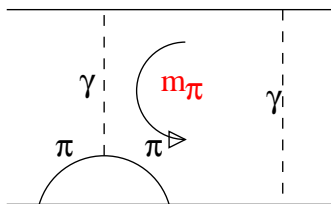
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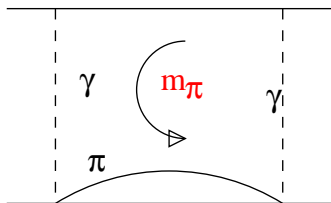
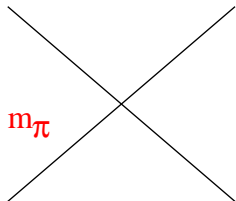
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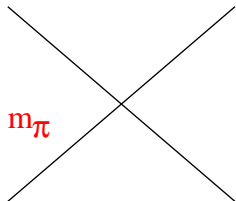
Leading chiral logs to the hyperfine splitting



$$\sim \frac{1}{f_\pi^2} \ln m_\pi$$



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$$\delta V = 2 \frac{C_{4,NR}}{m_p^2} \mathbf{S}^2 \delta^{(3)}(\mathbf{r}).$$

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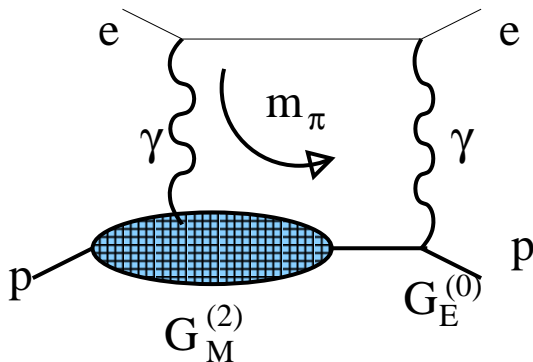


Figure: Symbolic representation (plus permutations) of the Zemach correction.

$$\delta c_{4,Zemach}^{pl} = (4\pi\alpha)^2 m_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{k^4} G_E^{(0)} G_M^{(2)}.$$

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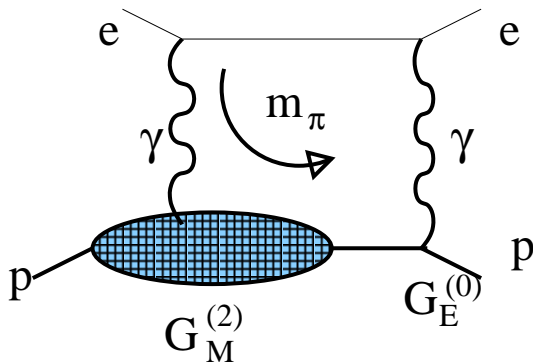


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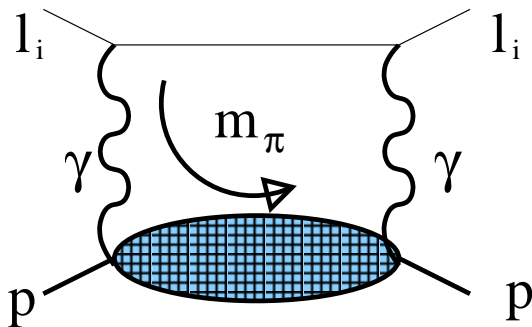


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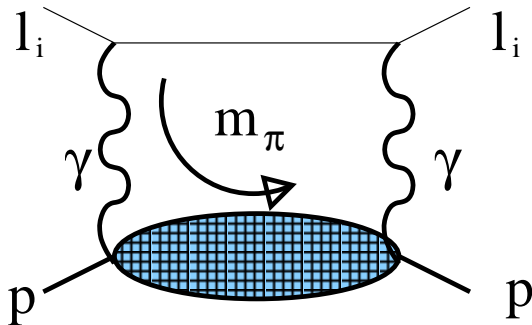


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$$T^{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle p, s | T J^\mu(x) J^\nu(0) | p, s \rangle ,$$

which has the following structure ($\rho = q \cdot p/m$):

$$\begin{aligned} T^{\mu\nu} = & \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) \\ & + \frac{1}{m_p^2} \left(p^\mu - \frac{m_p \rho}{q^2} q^\mu \right) \left(p^\nu - \frac{m_p \rho}{q^2} q^\nu \right) S_2(\rho, q^2) \\ & - \frac{i}{m_p} \epsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\rho, q^2) \\ & - \frac{i}{m_p^3} \epsilon^{\mu\nu\rho\sigma} q_\rho \left((m_p \rho) s_\sigma - (q \cdot s) p_\sigma \right) A_2(\rho, q^2) \end{aligned}$$

$$\delta C_{4,point-like}^{plj} = \frac{3 + 2c_F - c_F^2}{4} \alpha^2 \ln \frac{m_l^2}{\nu^2}.$$

$$\delta C_{4,Zemach-u,d}^{plj} \simeq \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \frac{2}{3} \pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2},$$

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$$\delta C_{4,pol.-\Delta}^{plj} = \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2},$$

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$$\delta C_{4,pol.-\pi\Delta}^{plj} = \frac{m_p^2}{(4\pi F_0)^2} g_{\pi N\Delta}^2 \frac{\alpha^2}{\pi} \frac{64}{27} C \ln \frac{\Delta^2}{\nu^2}.$$

Only logarithmically chiral enhanced but they can be determined from hydrogen hyperfine splitting.

$$\begin{aligned} \delta C_{4,NR}^{pl} &\simeq \left(1 - \frac{\mu_p^2}{4}\right) \alpha^2 \ln \frac{m_l^2}{\nu^2} \\ &+ \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \frac{2}{3} \left(\frac{2}{3} + \frac{7}{2\pi^2}\right) \pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2} \\ &+ \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \frac{8}{27} \left(\frac{5}{3} - \frac{7}{\pi^2}\right) \pi^2 g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2}, \end{aligned}$$

$$E_{\text{HF}} = 4 \frac{C_{4,NR}^{pl}}{m_p^2} \frac{1}{\pi} (\mu_l \rho \alpha)^3 \sim m_l \alpha^5 \frac{m_l^2}{m_p^2} \times (\ln m_q, \ln \Delta, \ln m_l).$$

Hydrogen. By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\text{HF,logarithms}}(m_\rho) = -0.031 \text{ MHz},$$

which accounts for approximately 2/3 of the difference between theory (pure QED) and experiment.

$$E_{\text{HF}}(\text{QED}) - E_{\text{HF}}(\text{exp}) = -0.046 \text{ MHz.}$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c_{4,NR}^{pl} = -47.7\alpha^2$ and $c_{4,R}^{pl}(m_\rho) \simeq c_{4,R}^D(m_\rho) \simeq -16\alpha^2$.

Muonic hydrogen.

$$\Delta E_{\text{HF}} \simeq -0.153 \text{ meV (Pachucki : -0.145)}$$

$$\Delta E = \frac{1}{4}(-0.15) \text{ meV} = -0.0375 \text{ meV}$$

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CONCLUSIONS

Important to have a **model independent** and **efficient** approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).

Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Theory appears to be solid, not to say extremely reliable. Only few a places where one should look "again" (out of desperation). Two/three-loop vacuum polarization potential? "Scheme" dependence? ...

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Beyond $m\alpha^5$. Resummation of logarithms

Pineda

$$\delta E_{n,l,j}^{\text{pot}}(\nu_{us}) = E_n \alpha^2 \frac{Z^2 \delta_{l0}}{3n} \left(-\frac{16}{\beta_0} \log \left(\frac{\alpha(\nu_{us})}{\alpha} \right) - 3(c_D - 1) \right),$$

where $c_D(\nu_s) = 1 + \frac{16}{3} \ln z$, $z = \left[\frac{\alpha(\nu_s)}{\alpha(m)} \right]^{\frac{1}{\beta_0}} \simeq 1 - 1/(2\pi)\alpha(\nu_s) \ln(\frac{\nu_s}{m})$,
 $\beta_0 = -\frac{4}{3} T_F n_f$ with $T_F = 1$, $E_n = -mZ^2 \alpha^2 / (2n^2)$, $\nu_s = 2a_n^{-1}$, where
 $a_n^{-1} = \frac{mZ\alpha(2a_n^{-1})}{n}$. α is also understood at the soft scale $\nu_s = 2a_n^{-1}$ unless the scale is specified. We take $\nu_{us} = -E_n$.

$$\begin{aligned} D_d^{(2)} - \frac{\alpha(\nu)}{2} &= \frac{\alpha(\nu_{us})}{2} \left(1 + \frac{\beta_0}{2\pi} \alpha(\nu_{us}) \ln \frac{mv^2}{mv} + \dots \right) \\ &\times \left(-\frac{8}{3} \frac{\alpha(\nu_{us})}{\pi} \ln \frac{mv^2}{m} - \frac{2}{3} \beta_0 \left(\frac{\alpha(\nu_{us})}{\pi} \right)^2 \ln^2 \frac{mv^2}{m} + \dots \right) \\ &\simeq -\frac{4}{3} \frac{\alpha^2(\nu_{us})}{\pi} \ln \frac{mv^2}{m} - \frac{2\beta_0}{3} \frac{\alpha^3(\nu_{us})}{\pi^2} \ln \frac{mv^2}{mv} \ln \frac{mv^2}{m} \\ &\quad - \frac{1}{3} \beta_0 \frac{\alpha^3(\nu_{us})}{\pi^2} \ln^2 \frac{mv^2}{m}. \end{aligned}$$

It agrees with Pachucki.