NON-RELATIVISTIC EFFECTIVE FIELD THEORIES:

RENORMALIZATION GROUP, RENORMALONS AND OBSERVABLES

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RENORMALIZATION GROUP IN

NON-RELATIVISTIC EFFECTIVE FIELD THEORIES

Motivation

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Resummation of logarithms in Quantum Field Theories (a long tale)

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Sudakov logarithms
Resummation of Logarithms in Deep Inelastic scattering
Resummation of logarithms in HQET
Electroweak logarithms
and so on ...

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Resummation of logarithms in Quantum Field Theories (a long tale)
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Electroweak logarithms
and so on ...

BUT!!!

WHAT ABOUT THE FIRST QUANTUM-FIELD-THEORY LOG?

THE LAMB SHIFT

$$\delta E \sim m\alpha^4 + m\alpha^5 \ln \alpha + (???) m\alpha^6 \ln^2 \alpha + \cdots$$

Summing logs in non-relativistic systems

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Large logs understood as ratios of scales: $\ln(mv/m) \sim \ln\alpha$**,** $\ln(mv^2/(mv)) \sim \ln\alpha$**.**

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\Gamma(V_Q(nS) \to e^+e^-) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \\
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Relevant for...

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Relevant for...
Determination of the bottom and charm masses

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Decays of heavy quarkonium
Heavy quarkonium sum rules

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t-$\bar{t}$ **production near threshold:** m_t, α_s, **Higgs-top coupling**

Renormalization group in NRQCD (LL) (Soft running)

Aim: **to obtain the running of the NRQCD matching coefficients:** $\left(\alpha_s \ln \frac{m}{\nu}\right)^n$

Relevant for:

- **pNRQCD in the perturbative regime**
- **pNRQCD in the nonperturbative regime**
- **"Standard" NRQCD**

$$
\begin{aligned}
\mathcal{L}_{NRQCD} &= \bar{\Psi} i\gamma^0 D_0 \Psi + \bar{\Psi}\{\frac{\mathbf{D}^2}{2m} + c_F\, g\frac{\mathbf{\Sigma}\cdot\mathbf{B}}{2m} + c_D\, g\frac{\gamma^0\left(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D}\right)}{8m^2} \\
&+ i c_S\, g\frac{\gamma^0\mathbf{\Sigma}\cdot\left(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}\right)}{8m^2} + \frac{\mathbf{D}^4}{8m^3}\}\Psi \\
&- \frac{1}{4}c_1 F_{\mu\nu}F^{\mu\nu} + \frac{c_2}{m^2} g F_{\mu\nu} D^2 g F^{\mu\nu} + \frac{c_3}{m^2} g^3 f_{ABC} F^A_{\mu\nu} F^B_{\mu\alpha} F^C_{\nu\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\delta\mathcal{L}_{NRQCD} &= \frac{d_{ss}}{m_1 m_2}\psi_1^\dagger\psi_1\chi_2^\dagger\chi_2 + \frac{d_{sv}}{m_1 m_2}\psi_1^\dagger\boldsymbol{\sigma}\psi_1\chi_2^\dagger\boldsymbol{\sigma}\chi_2 \\
&+ \frac{d_{vs}}{m_1 m_2}\psi_1^\dagger \mathrm{T}^a\psi_1\chi_2^\dagger \mathrm{T}^a\chi_2 + \frac{d_{vv}}{m_1 m_2}\psi_1^\dagger \mathrm{T}^a\boldsymbol{\sigma}\psi_1\chi_2^\dagger \mathrm{T}^a\boldsymbol{\sigma}\chi_2 \,.
\end{aligned}
$$

Typically, $\quad c_i \sim 1 + \sum_n A_n \left(\alpha_s \ln \frac{m}{\nu}\right)^n \qquad d_i \sim \alpha_s\left(1 + \sum_n B_n \left(\alpha_s \ln \frac{m}{\nu}\right)^n\right)$

$\nu_p \gg |\mathbf{p}|$: **quark-antiquark relative three-momentum.**

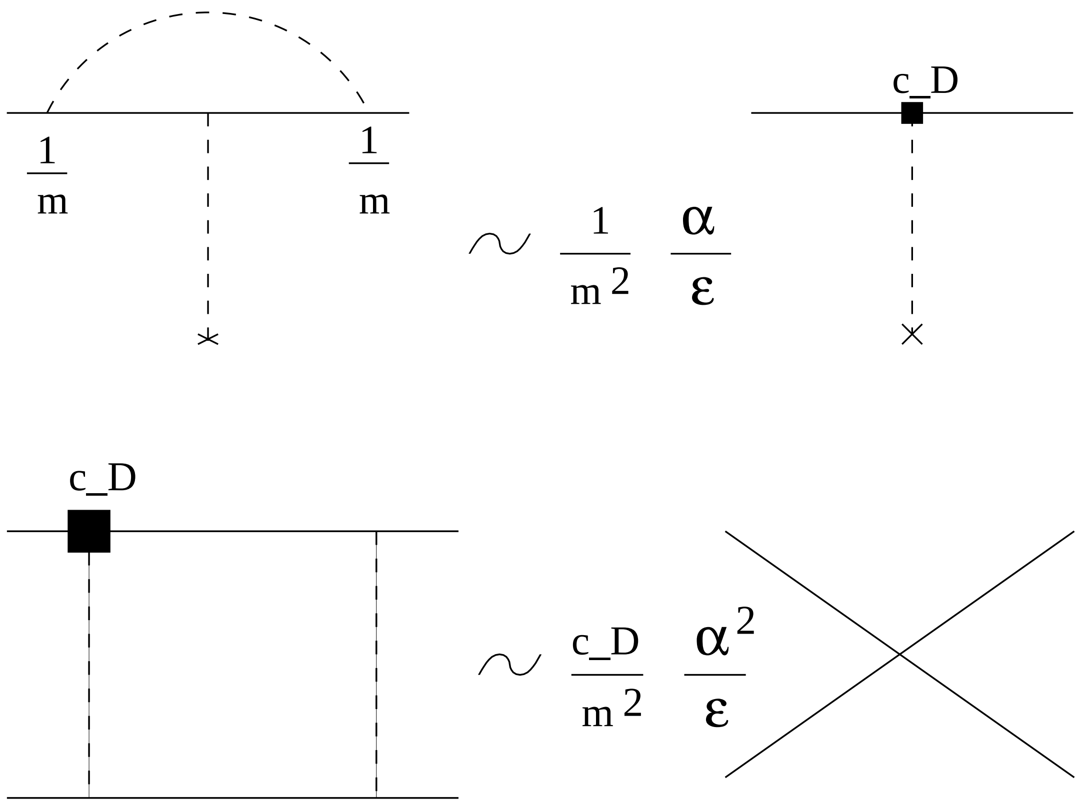
$\nu_s \gg |\mathbf{k}|$: **gluon three-momentum, transfer momentum between the quark and antiquark.**

$m \gg \nu_p \sim \nu_s$

Matching coefficients: $c(\nu_s)$, $d(\nu_s, \nu_p)$

LL $\rightarrow$ $c(\nu_s)$, $d(\nu_s)$

Running ν_s **LL: HQET;** $1/m$ **expansion,** $\dfrac{i}{q^0 + i\epsilon}$



$$\nu_s \frac{d}{d\nu_s} c_D = \frac{\alpha_s}{4\pi}\left[\frac{4C_A}{3}c_D - \left(\frac{2C_A}{3}+\frac{32C_f}{3}\right)c_k^2 - \frac{10C_A}{3}c_F^2 + \frac{8T_F n_f}{3}c_1^{hl}\right],$$

$$\begin{aligned}
\nu_s \frac{d}{d\nu_s} d_{ss} &= -2C_f\left(C_f - \frac{C_A}{2}\right)\alpha_s^2 c_k^2\,, \\
\nu_s \frac{d}{d\nu_s} d_{sv} &= 0\,, \\
\nu_s \frac{d}{d\nu_s} d_{vs} &= 4\left(C_f - C_A\right)\alpha_s^2 c_k^2 + \frac{3}{2}\alpha_s^2 C_A c_D\,, \\
\nu_s \frac{d}{d\nu_s} d_{vv} &= -\frac{C_A}{2}\alpha_s^2 c_F^2\,.
\end{aligned}$$

We define $z = \left[\frac{\alpha_s(\nu_s)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}} \simeq 1 - 1/(2\pi)\alpha_s(\nu_s)\ln(\frac{\nu_s}{m})$**,** $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$

$$\begin{aligned}
c_F(\nu_s) &= z^{-C_A}\,, \\
c_S(\nu_s) &= 2z^{-C_A} - 1\,, \\
c_D(\nu_s) &= \frac{9C_A}{9C_A + 8T_F n_f}\left\{-\frac{5C_A + 4T_F n_f}{4C_A + 4T_F n_f}z^{-2C_A} + \frac{C_A + 16C_f - 8T_F n_f}{2(C_A - 2T_F n_f)}\right. \\
&\quad + \frac{-7C_A^2 + 32C_A C_f - 4C_A T_F n_f + 32C_f T_F n_f}{4(C_A + T_F n_f)(2T_F n_f - C_A)} z^{4T_F n_f/3 - 2C_A/3} \\
&\quad \left. + \frac{8T_F n_f}{9C_A}\left[z^{-2C_A} + \left(\frac{20}{13} + \frac{32}{13}\frac{C_f}{C_A}\right)\left[1 - z^{\frac{-13C_A}{6}}\right]\right]\right\},
\end{aligned}$$

$$
\begin{aligned}
d_{ss}(\nu_s) &= d_{ss}(m) + 4C_f\left(C_f - \frac{C_A}{2}\right)\frac{\pi}{\beta_0}\alpha_s(m)\left[z^{\beta_0} - 1\right] , \\
d_{sv}(\nu_s) &= d_{sv}(m) , \\
d_{vs}(\nu_s) &= d_{vs}(m) - (C_f - C_A)\frac{8\pi}{\beta_0}\alpha_s(m)\left[z^{\beta_0} - 1\right] \\
&\quad - \frac{27C_A^2}{9C_A + 8T_F n_f}\frac{\pi}{\beta_0}\alpha_s(m)\left\{-\frac{5C_A + 4T_F n_f}{4C_A + 4T_F n_f}\frac{\beta_0}{\beta_0 - 2C_A}\left(z^{\beta_0 - 2C_A} - 1\right)\right. \\
&\qquad + \frac{C_A + 16C_f - 8T_F n_f}{2(C_A - 2T_F n_f)}\left(z^{\beta_0} - 1\right) \\
&\qquad + \frac{-7C_A^2 + 32C_A C_f - 4C_A T_F n_f + 32C_f T_F n_f}{4(C_A + T_F n_f)(2T_F n_f - C_A)} \\
&\qquad\qquad \times \frac{3\beta_0}{3\beta_0 + 4T_F n_f - 2C_A}\left(z^{\beta_0 + 4T_F n_f/3 - 2C_A/3} - 1\right) \\
&\qquad + \frac{8T_F n_f}{9C_A}\left[\frac{\beta_0}{\beta_0 - 2C_A}\left(z^{\beta_0 - 2C_A} - 1\right) + \left(\frac{20}{13} + \frac{32}{13}\frac{C_f}{C_A}\right)\right. \\
&\qquad\qquad \left.\left.\times\left(\left[z^{\beta_0} - 1\right] - \frac{6\beta_0}{6\beta_0 - 13C_A}\left[z^{\beta_0 - \frac{13C_A}{6}} - 1\right]\right)\right]\right\} , \\
d_{vv}(\nu_s) &= d_{vv}(m) + \frac{C_A}{\beta_0 - 2C_A}\pi\alpha_s(m)\left\{z^{\beta_0 - 2C_A} - 1\right\} .
\end{aligned}
$$

Bauer-Manohar; Pineda

One equation for the soft running.

Renormalization group in pNRQCD (LL) (Ultrasoft running)

Aim: to obtain the running of the pNRQCD matching coefficients: $(\alpha_s \ln)^n$, $\alpha_s (\alpha_s \ln)^n$

Relevant for:

- **Spectrum: heavy quarkonium and QED.**
- **Currents: electromagnetic decays.**
- **Currents: Normalization of bottomonium sum rules.**
- **Currents: Normalization of t-$\bar{t}$ production near threshold.**

$$L_{pNRQCD} = L'_{NRQCD} + \int\int d^3x_1 d^3x_2 \psi(x_1)\chi_c(x_2) V(x_1 - x_2)\psi^\dagger(x_1)\chi_c^\dagger(x_2)$$

L'_{NRQCD}, **gluons multipole expanded (only ultrasoft gluons).**

$$\begin{aligned}
\mathcal{L}_{pNRQCD} &= \mathrm{Tr}\Big\{S^\dagger \left(i\partial_0 - V_s^{(0)}(\mathbf{x})\right) S + O^\dagger \left(iD_0 - V_o^{(0)}(\mathbf{x})\right) O\Big\} \\
&+ gV_A(\mathbf{x})\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, S + S^\dagger \mathbf{x}\cdot\mathbf{E}\, O\right\} + g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, O + O^\dagger O \mathbf{x}\cdot\mathbf{E}\right\} \\
&- \mathrm{Tr}\Big\{S^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}(\mathbf{x})}{m^n}\right) S - O^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_o^{(n)}(\mathbf{x})}{m^n}\right) O\Big\},
\end{aligned}$$

$$V_s^{(0)} \equiv -C_F \frac{\alpha_{V_s}}{r}.$$

$$\frac{V_s^{(1)}}{m} \equiv -\frac{C_F C_A D_s^{(1)}}{2mr^2}.$$

$$\begin{aligned}
\frac{V_s^{(2)}}{m^2} &= -\frac{C_F D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r}, \mathbf{p}^2\right\} + \frac{C_F D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2 + \frac{\pi C_F D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r}) \\
&+ \frac{4\pi C_F D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_F D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_F D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),
\end{aligned}$$

where $S_{12}(\hat{\mathbf{r}}) \equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\, \hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ **and** $\mathbf{S} = \boldsymbol{\sigma}_1/2 + \boldsymbol{\sigma}_2/2$**.**

$\nu_p \gg |\mathbf{p}|$: **quark-antiquark relative three-momentum.**

$\nu_{us} \gg |\mathbf{k}|$: **gluon three-momentum.**

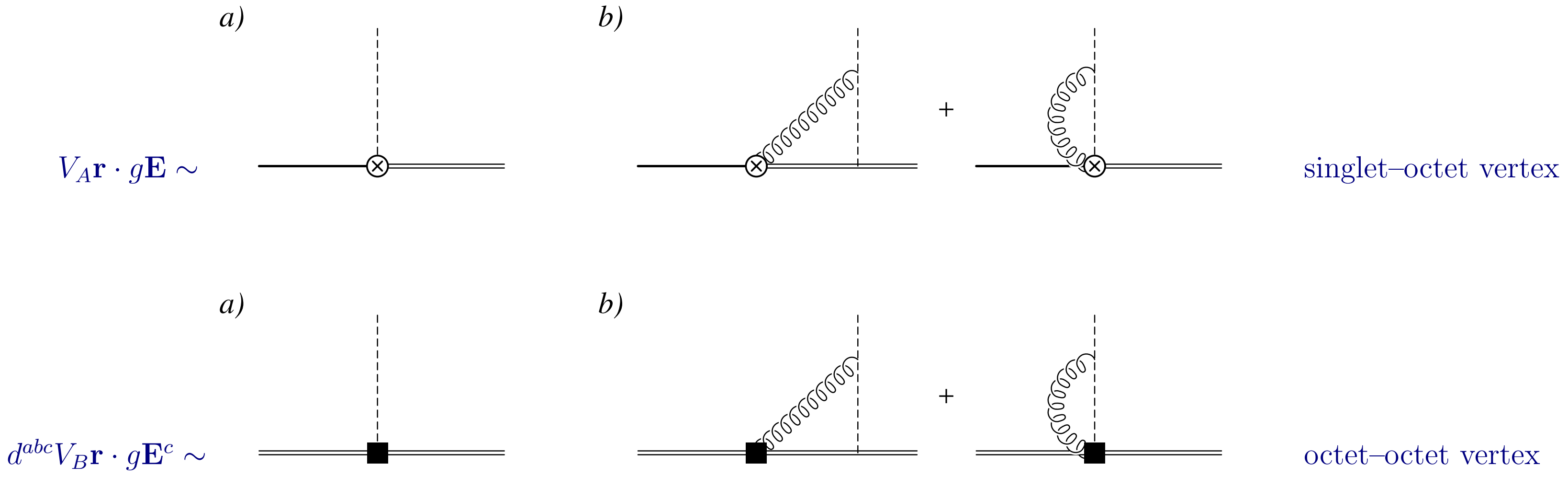
$|\mathbf{p}| \gg \nu_{us} \gg \mathbf{p}^2/m$

Matching coefficients: $\tilde{V}(d(\nu_p,\nu_s,m),c(\nu_s,m),\nu_s,\nu_{us},r) = \tilde{V}(\nu_p,m,\nu_{us},r) \equiv \tilde{V}(\nu_p,\nu_{us})\,.$

$\nu_s \frac{d}{d\nu_s}\tilde{V} = 0$; $\nu_s = 1/r$

LL: $\nu_p \frac{d}{d\nu_p}\tilde{V} = 0$

ν_{us}. **The computation can be formally organized through the multipole expansion.**



Corrections to the Green Function

$$G_s(E) = P_s \frac{1}{H - H_I - E} P_s = G_s^{(0)} + \delta G_s$$

From the potential:

$$\delta G_s \sim \frac{1}{H_s - E} \delta V \frac{1}{H_s - E}$$

From ultrasoft gluons:

$$\begin{aligned} \delta G_s \;&\sim\; \frac{1}{H_s - E} \int \frac{d^3\mathbf{k}}{(2\pi)^{D-1}} \mathbf{r} \frac{k}{k + H_o - E} \mathbf{r} \frac{1}{H_s - E} \\ &\sim\; \frac{1}{H_s - E} \mathbf{r} (H_o - E)^3 \left\{ \frac{1}{\epsilon} + \gamma + \ln \frac{(H_o - E)^2}{\nu^2} + C \right\} \mathbf{r} \frac{1}{H_s - E} \end{aligned}$$

$$
\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_s} &= \frac{2}{3}\frac{\alpha_s}{\pi}V_A^2\left(\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right)^3,\\
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_o} &= \frac{2}{3}\frac{\alpha_s}{\pi}V_A^2\left(\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right)^3,\\
\nu_{us}\frac{d}{d\nu_{us}}\alpha_s &= -\beta_0\frac{\alpha_s^2}{2\pi},\\
\nu_{us}\frac{d}{d\nu_{us}}V_A &= 0,\\
\nu_{us}\frac{d}{d\nu_{us}}V_B &= 0\,.
\end{aligned}
$$

$$
\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}C_A D_s^{(1)} &= \frac{16}{3}\frac{\alpha_s}{\pi}V_A^2 c_k\left[\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right]\left[2C_f\alpha_{V_s}+\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}\right],\\
\nu_{us}\frac{d}{d\nu_{us}}D_{d,s}^{(2)} &= \frac{16}{3}\frac{\alpha_s}{\pi}V_A^2 c_k^2\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o},\\
\nu_{us}\frac{d}{d\nu_{us}}D_{1,s}^{(2)} &= \frac{8}{3}\frac{\alpha_s}{\pi}V_A^2 c_k^2\left[\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right],
\end{aligned}
$$

and zero for the other matching coefficients (in particular for the spin-dependent potentials).

Soto-Pineda; Pineda

RG equations within an strict expansion in α

$$\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_s} &= \frac{2}{3}\frac{\alpha_s(\nu_{us})}{\pi}\left(\frac{C_A}{2}\right)^3\alpha_s^3(r^{-1})\,,\\
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_o} &= 0\,,\\
\nu_{us}\frac{d}{d\nu_{us}}C_A D_s^{(1)} &= \frac{16}{3}\frac{\alpha_s(\nu_{us})}{\pi}\frac{C_A}{2}\left(C_f+\frac{C_A}{2}\right)\alpha_s^2(r^{-1})\,,\\
\nu_{us}\frac{d}{d\nu_{us}}D_{1,s}^{(2)} &= \frac{8}{3}\frac{\alpha_s(\nu_{us})}{\pi}\frac{C_A}{2}\alpha_s(r^{-1})\,,\\
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\end{aligned}$$

Initial conditions ($\nu_{us} = 1/r$):

$$
\begin{aligned}
\alpha_{V_s}(r^{-1}) &= \alpha_{\rm s}(r^{-1})\left\{1 + (a_1 + 2\gamma_E\beta_0)\,\frac{\alpha_{\rm s}(r^{-1})}{4\pi}\right. \\
&\quad \left. + \left[\gamma_E\,(4a_1\beta_0 + 2\beta_1) + \left(\frac{\pi^2}{3} + 4\gamma_E^2\right)\beta_0^2 + a_2\right]\frac{\alpha_{\rm s}^2(r^{-1})}{16\,\pi^2}\right\}, \\
D_s^{(1)}(r^{-1}) &= \alpha_{\rm s}^2(r^{-1}), \\
D_{1,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1}), \\
D_{2,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1}), \\
D_{d,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1})(2 + c_D(r^{-1}) - 2c_F^2(r^{-1})) \\
&\quad + \frac{1}{\pi}\left[d_{vs}(r^{-1}) + 3d_{vv}(r^{-1}) + \frac{1}{C_f}(d_{ss}(r^{-1}) + 3d_{sv}(r^{-1}))\right], \\
D_{S^2,s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1})c_F^2(r^{-1}) - \frac{3}{2\pi C_f}(d_{sv}(r^{-1}) + C_f d_{vv}(r^{-1})), \\
D_{LS,s}^{(2)}(r^{-1}) &= \frac{\alpha_{\rm s}(r^{-1})}{3}(c_S(r^{-1}) + 2c_F(r^{-1})), \\
D_{S_{12},s}^{(2)}(r^{-1}) &= \alpha_{\rm s}(r^{-1})c_F^2(r^{-1}), \\
\alpha_{V_o}(r^{-1}) &= \alpha_{\rm s}(r^{-1}), \\
V_A(r^{-1}) &= 1,
\end{aligned}
$$

The RG improved potentials for the singlet read:

$$
\begin{aligned}
\alpha_{V_s}(\nu_{us}) &= \alpha_{V_s}(r^{-1}) + \frac{C_A^3}{6\beta_0}\alpha_s^3(r^{-1}) \log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_s^{(1)}(\nu_{us}) &= D_s^{(1)}(r^{-1}) + \frac{16}{3\beta_0}\left(\frac{C_A}{2} + C_f\right)\alpha_{\rm s}^2(r^{-1}) \log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_{1,s}^{(2)}(\nu_{us}) &= D_{1,s}^{(2)}(r^{-1}) + \frac{8C_A}{3\beta_0}\alpha_{\rm s}(r^{-1}) \log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_{2,s}^{(2)}(\nu_{us}) &= D_{2,s}^{(2)}(r^{-1}), \\
D_{d,s}^{(2)}(\nu_{us}) &= D_{d,s}^{(2)}(r^{-1}) + \frac{32}{3\beta_0}\left(\frac{C_A}{2} - C_f\right)\alpha_s(r^{-1}) \log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_{S^2,s}^{(2)}(\nu_{us}) &= D_{S^2,s}^{(2)}(r^{-1}), \\
D_{LS,s}^{(2)}(\nu_{us}) &= D_{LS,s}^{(2)}(r^{-1}), \\
D_{S_{12},s}^{(2)}(\nu_{us}) &= D_{S_{12},s}^{(2)}(r^{-1}).
\end{aligned}
$$

Soto, Pineda; Pineda

One equation for the ultrasoft running.

OBSERVABLE: NNLL heavy quarkonium mass $O(m\alpha^{4+n}\ln^n\alpha)$ (Pineda; Hoang-Stewart)

$$\delta E^{\rm pot}_{n,l,j}(\nu_{us}) = E_n\alpha_s^2\left\{-\frac{2C_A}{3\beta_0}\left[\frac{C_A^2}{2}+4C_AC_f\frac{1}{n(2l+1)}+2C_f^2\left(\frac{8}{n(2l+1)}-\frac{1}{n^2}\right)\right]\log\left(\frac{\alpha_s(\nu_{us})}{\alpha_s}\right)\right.$$
$$+\frac{C_f^2\delta_{l0}}{3n}\left(-\frac{16}{\beta_0}\left[C_f-\frac{C_A}{2}\right]\log\left(\frac{\alpha_s(\nu_{us})}{\alpha_s}\right)\right.$$
$$\left.-\frac{3}{2}(1+c_D-2c_F^2)-\frac{3}{2\pi\alpha_s}\left[d_{vs}+3d_{vv}+\frac{1}{C_f}(d_{ss}+3d_{sv})\right]\right)$$
$$-\frac{4}{3}\frac{C_f^2\delta_{l0}\delta_{s1}}{n}\left\{z^{-2C_A}-1+\frac{3}{2}\frac{C_A}{\beta_0-2C_A}\left[z^{-\beta_0}-z^{-2C_A}\right]\right\}$$
$$\left.-\frac{(1-\delta_{l0})\delta_{s1}}{l(2l+1)(l+1)n}C_{j,l}\frac{C_f^2}{2}\right\},$$

where $E_n=-mC_f^2\alpha_s^2/(4n^2)$, $\nu_s=2a_n^{-1}$ **where** $2a_n^{-1}=\frac{mC_f\alpha_s(2a_n^{-1})}{n}$, **and**

$$C_{j,l}=\begin{cases}-\frac{(l+1)}{2\,l-1}\left\{4(2l-1)\left(z^{-C_A}-1\right)+\left(z^{-2C_A}-1\right)\right\} & ,\ j=l-1\\ -4\left(z^{-C_A}-1\right)+\left(z^{-2C_A}-1\right) & ,\ j=l\\ \frac{l}{2\,l+3}\left\{4(2l+3)\left(z^{-C_A}-1\right)-\left(z^{-2C_A}-1\right)\right\} & ,\ j=l+1.\end{cases}$$

Check with $O(m\alpha^5\ln\alpha)$ known logs: Brambilla, Vairo, Soto, Pineda; Kniehl, Penin; Hoang, Manohar and Stewart.

Muonic Hydrogen mass at NNLL. Check with known logs by Pachucki.

Renormalization group in pNRQCD (NLL) (potential running)

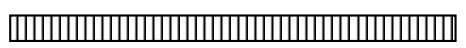
ν_p **enters into the game.**
The running on ν_p can be obtained from pNRQCD (there is no running of ν_p from NRQCD to pNRQCD). It can be obtained by Quantum mechanics computations. Example: iteration of the potentials. Divergent integrals in $|\mathbf{p}|$ and r.

$$h_s = c_k\frac{\mathbf{p}^2}{m} - C_f\frac{\alpha_{V_s}}{r} - c_4\frac{\mathbf{p}^4}{4m^3} - \frac{C_fC_AD_s^{(1)}}{2mr^2} - \frac{C_fD_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r},\mathbf{p}^2\right\} + \frac{C_fD_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2$$
$$+\frac{\pi C_fD_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r}) + \frac{4\pi C_fD_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_fD_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_fD_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}})\,,$$

where $C_f = (N_c^2-1)/(2N_c)$ and $c_k = c_4 = 1$ (we only use c_4 for tracking of the contribution due to this term). The propagator of the singlet is (formally)

$$\frac{1}{E-h_s}\,.$$

At leading order (within an strict expansion in α_s) the propagator of the singlet reads

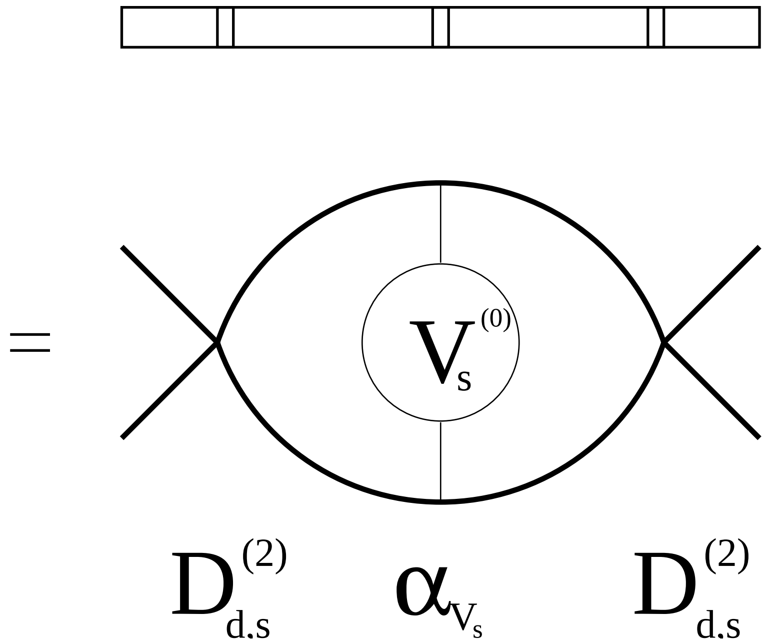
 $$= G_c(E) = \frac{1}{E-h_s^{(0)}} = \frac{1}{E-\mathbf{p}^2/m - C_f\alpha_s/r}\,.$$

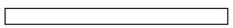
If we were interested in computing the spectrum at $O(m\alpha_s^6)$, one should consider the iteration of subleading potentials (δh_s) in the propagator as

follows:

$$G_c(E)\delta h_s G_c(E)\cdots\delta h_s G_c(E)\,.$$

In general, these contributions will produce logarithmic divergences due to potential loops. These divergences can be absorbed in the matching coefficients, $D^{(2)}_{d,s}$ and $D^{(2)}_{S^2,s}$, of the local potentials (proportional to the $\delta^{(3)}(\mathbf{r})$) providing with the renormalization group equations of these matching coefficients in terms of ν_p. Let us explain how it works in detail. Since the singular behavior of the potential loops appears for $\mathbf{p}^2/m \gg \alpha_s/r$, a perturbative expansion in α_s is licit in $G_c(E)$, which can be approximated by



 $= G_c^{(0)}(E) = \dfrac{1}{E-\mathbf{p}^2/m}\,.$

$$\langle \mathbf{r}=0|\frac{1}{E-\mathbf{p}^2/m}C_f\frac{\alpha_{V_s}}{r}\frac{1}{E-\mathbf{p}^2/m}|\mathbf{r}=0\rangle$$
$$\sim \int \frac{\mathrm{d}^d p'}{(2\pi)^d}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{m}{\mathbf{p'}^2-mE}C_f\frac{4\pi\alpha_{V_s}}{\mathbf{q}^2}\frac{m}{\mathbf{p}^2-mE} \sim -C_f\frac{m^2\alpha_{V_s}}{16\pi}\frac{1}{\epsilon},$$

where $D = 4+2\epsilon$ **and** $\mathbf{q}=\mathbf{p}-\mathbf{p}'$**. This divergence is absorbed in** $D^{(2)}_{d,s}$ **contributing to its running at NLL order as follows**

$$\nu_p\frac{d}{d\nu_p}D^{(2)}_{d,s}(\nu_p) \sim \alpha_{V_s}(\nu_p)D^{(2)2}_{d,s}(\nu_p)+\cdots .$$

$O(m\alpha^8\ln^3\alpha)$ **correction to the Hydrogen atom spectrum. Manohar-Stewart; Pineda**

and one equation for the potential running

$|\mathbf{p}| \gg \nu_{us} \gg \mathbf{p}^2/m \to \nu_{us} = {\nu_p}^2/m$ **(Luke, Manohar, Rothstein)**

We can not lower ν_{us} further. Fight between two terms.

$$\frac{1}{\mathbf{p}^2/m + k}$$

$$\begin{aligned}\tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_p^2/m, r) \;\simeq\; & \tilde{V}(c(\nu_p), d(\nu_p, \nu_p), \nu_p, \nu_p^2/m, \nu_p) \\ & + \ln(\nu_p r) r \frac{d}{dr} \tilde{V}|_{1/r=\nu_p} + \cdots \,.\end{aligned}$$

One equation for the soft running,
one equation for the ultrasoft running,
and one equation for the potential running,
which rules them all and at the hard scale binds them.

Nonrelativistic Sum rules (b-$\bar{b}$, c-$\bar{c}$), t-$\bar{t}$ production near threshold

Determination of m_b, m_t, α_s, Higgs-top yukawa coupling, . . .

$$J^\mu = \bar{Q}\gamma^\mu Q = B_1 \psi^\dagger \boldsymbol{\sigma} \chi + \cdots ,$$

$$B_1 = 1 + a_1\alpha_s + a_2\alpha_s^2 + \cdots$$

B_1 at NNLO: Hoang(QED); Beneke, Signer, Smirnov; Czarnecki, Melnikov
B_1, B_0 at NLL: Pineda; Hoang, Stewart
B_1/B_0 at NNLL: Penin, Pineda, Smirnov, Steinhauser
B_1, B_0 at NNLL (partial): Pineda, Signer

$$(q_\mu q_\nu - g_{\mu\nu})\Pi(q^2) = i\int d^4x e^{iqx}\langle \mathrm{vac}|J_\mu(x)J_\nu(0)|\mathrm{vac}\rangle$$

$$\Pi(q^2) \sim B_1^2 \langle \mathbf{r}=\mathbf{0}|\frac{1}{E-H}|\mathbf{r}=\mathbf{0}\rangle$$

$$G(0,0,E) = \sum_{m=0}^{\infty}\frac{|\phi_{0m}(0)|^2}{E_{0m}-E+i\epsilon-i\Gamma_t} + \frac{1}{\pi}\int_0^\infty dE' \frac{|\phi_{0E'}(0)|^2}{E_{0E'}-E+i\epsilon-i\Gamma_t}$$

A NNLL renormalization group improved expression of $M(V_Q(nS))$ is also needed in order to obtain expressions for the t-$\bar{t}$ production near threshold with NNLL accuracy:

$M(V_Q(nS))$ at NNLL: Pineda; Hoang, Stewart
$M(V_Q(nS)) - M(P_Q(nS))$ at NNNLL: Kniehl, Penin, Pineda, Smirnov, Steinhauser

Relation of the vacuum polarization with $\sigma_{t\bar{t}}$, non-relativistic sum rules and $\Gamma(V_Q(nS) \to e^+e^-)$

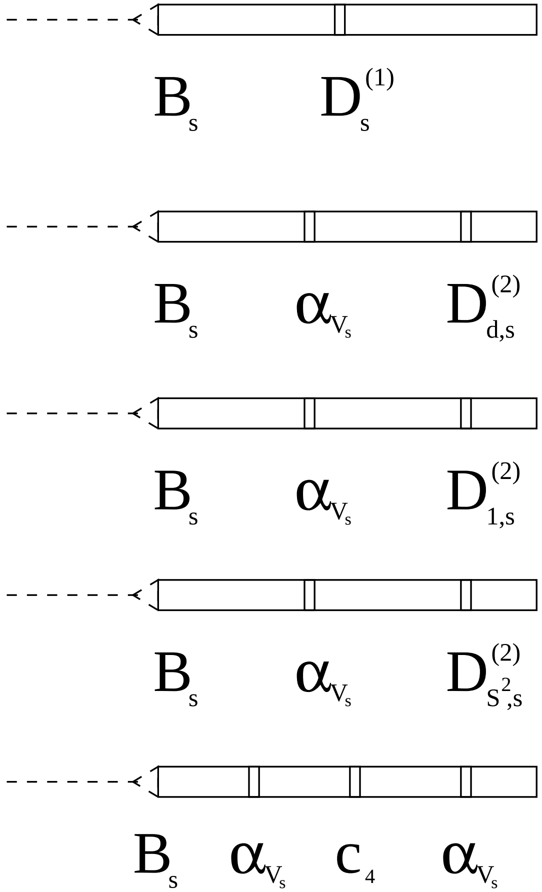
$$\Gamma(V \to e^+e^-) \sim \frac{1}{m^2} B_1^2 |\phi(\mathbf{0})|^2$$

$$\sigma_{t-\bar{t}} \sim B_1(\nu)^2 \mathrm{Im} G(0,0,\sqrt{s}) + \cdots$$

$$M_n \equiv \frac{12\pi^2 e_b^2}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2)|_{q^2=0} = \int_0^\infty \frac{ds}{s^{n+1}} R_{b\bar{b}}(s),$$

$$M_n = 48\pi e_b^2 N_c \int_{-\infty}^{\infty} \frac{dE}{(E+2m_b)^{2n+3}} \left(B_1^2 - B_1 d_1 \frac{E}{3m_b}\right) \mathrm{Im}\, G(0,0,E)$$

Matching coefficient of the electromagnetic current at NLL



$$\nu_p \frac{d}{d\nu_p} B_s = -\frac{C_A C_f}{2} D_s^{(1)} - \frac{C_f^2}{4} \alpha_s \left\{ \alpha_s - \frac{4}{3} s(s+1) D_{S^2,s}^{(2)} - D_{d,s}^{(2)} + 4 D_{1,s}^{(2)} \right\} ,$$

$$b_1(m) = 1 - 2C_f \frac{\alpha_s(m)}{\pi}, \qquad b_0(m) = 1 + \left(\frac{\pi^2}{4} - 5 \right) \frac{C_f}{2} \frac{\alpha_s(m)}{\pi}.$$

The solution reads (Pineda; Hoang-Stewart)

$$B_s(\nu_p) = b_s(m) + A_1\frac{\alpha_s(m)}{w^{\beta_0}}\ln(w^{\beta_0}) + A_2\alpha_s(m)[z^{\beta_0}-1] + A_3\alpha_s(m)[z^{\beta_0-2C_A}-1]$$
$$+A_4\alpha_s(m)[z^{\beta_0-13C_A/6}-1] + A_5\alpha_s(m)\ln(z^{\beta_0}),$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$, $z = \left[\frac{\alpha_s(\nu_p)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}}$ **and** $w = \left[\frac{\alpha_s(\nu_p^2/m)}{\alpha_s(\nu_p)}\right]^{\frac{1}{\beta_0}}$**. The coefficients** A_i **read**

$$A_1 = \frac{8\pi C_f}{3\beta_0^2}\left(C_A^2+2C_f^2+3C_fC_A\right),$$
$$A_2 = \frac{\pi C_f[3\beta_0(26C_A^2+19C_AC_f-32C_f^2)-C_A(208C_A^2+651C_AC_f+116C_f^2)]}{78\,\beta_0^2\,C_A},$$
$$A_3 = -\frac{\pi C_f^2[\beta_0(4s(s+1)-3)+C_A(15-14s(s+1))]}{6(\beta_0-2C_A)^2},$$
$$A_4 = \frac{24\pi C_f^2(3\beta_0-11C_A)(5C_A+8C_f)}{13\,C_A(6\beta_0-13C_A)^2},$$
$$A_5 = \frac{-\pi C_f^2}{\beta_0^2\,(6\beta_0-13C_A)(\beta_0-2C_A)}\{C_A^2(-9C_A+100C_f)$$
$$+\beta_0C_A(-74C_f+C_A(42-13s(s+1)))+6\beta_0^2(2C_f+C_A(-3+s(s+1)))\}.$$

Leading (Czarnecki-Melnikov; Beneke-Signer-Smirnov) and subleading (Kniehl-Penin) logs correct.

Inclusive decays to leptons and photons at NLL (Pineda)

By setting $\nu_p \sim m\alpha_s$, $B_s(\nu_p)$ includes all the large logs at **NLL** order in any (inclusive enough) S-wave heavy-quarkonium production observable we can think of. For instance, the decays to e^+e^- and to **two photons** at **NLL** $O(\alpha^{1+n}\ln^n\alpha)$ order read

$$\begin{aligned}
\Gamma(V_Q(nS) \to e^+e^-) &= 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2 \left(\frac{m_Q C_f \alpha_s}{n}\right)^3 \left\{B_1(\nu_p)(1+\delta\phi_n)\right\}^2 \\
&\simeq 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2 \left(\frac{m_Q C_f \alpha_s}{n}\right)^3 \left\{1+2(B_1(\nu_p)-1)+2\delta\phi_n\right\}, \\
\Gamma(P_Q(nS) \to \gamma\gamma) &= 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2 \left(\frac{m_Q C_f \alpha_s}{n}\right)^3 \left\{B_0(\nu_p)(1+\delta\phi_n)\right\}^2 \\
&\simeq 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2 \left(\frac{m_Q C_f \alpha_s}{n}\right)^3 \left\{1+2(B_0(\nu_p)-1)+2\delta\phi_n\right\},
\end{aligned}$$

where V and P stand for the vector and pseudoscalar heavy quarkonium, we have fixed $\nu_p = m_Q C_f \alpha_s/n$, $\alpha_s = \alpha_s(\nu_p)$, and ($\Psi_n(z) = \frac{d^n \ln\Gamma(z)}{dz^n}$ and $\Gamma(z)$ is the Euler Γ-function)

$$\delta\phi_n = \frac{\alpha_s}{\pi}\left[-C_A + \frac{\beta_0}{4}\left(\Psi_1(n+1) - 2n\Psi_2(n) + \frac{3}{2} + \gamma_E + \frac{2}{n}\right)\right].$$

NONRELATIVISTIC EFFECTIVE FIELD THEORIES

AND

RENORMALONS

Question

$$M_{\Upsilon(1S)} = m_{\rm OS}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \cdots)$$

What if $A_n \sim n!$? Bad convergence
Should we expect that?

Renormalons

They are a potential problem in effective field theories of QCD (OPE) where the matching coefficients can be computed in perturbation theory.

Examples:

OPE
NRQCD
HQET
pNRQCD
SCET

Renormalons appear as soon as we have factorization between different scales: They can deteriorate the convergence of the perturbative series in QCD.

Can one understand the renormalon within an effective field theory/factorization formalism?

Problems:

1) Fix the parameters of the Standard Model. Search for weakly sensitive to long distance physics observables. One wants to avoid spurious dependence on the renormalon.

2) Meaningful determination of non-perturbative parameters.

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n$$

Matching coefficients suffer from renormalon ambiguities that cancel with the ones of the matrix elements in effective field theory calculations.

$$c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

Its Borel transform would be

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^{\infty} \mathbf{d}t \, e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the matching coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

$$\delta B[c](t) \sim \frac{1}{a-t},$$

where a is a positive number, it sets up the maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation, which is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*},$$

where $n^* \sim \frac{a}{\alpha_s}$. Moreover, the fact that a is positive means that, even after Borel resummation, c suffers from a non-perturbative ambiguity of order

$$\delta c \sim \left(\Lambda_{QCD}\right)^{\frac{a\beta_0}{2\pi}} \ .$$

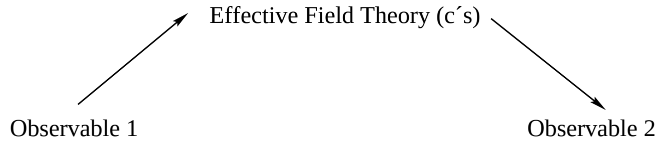


Figure 1: *Symbolic relation between observables through the determination of the matching coefficients of the effective field theory.*

Examples

$$M_B = m_{\rm OS} + \bar{\Lambda} + \mathcal{O}(1/m_{\rm OS})$$

M_B **is renormalon free. Problem. Therefore** $m_{\rm OS}$ **suffers from renormalon ambiguities:**

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots)$$

with $B_n \sim n!$. **In other words**

$$\delta_{np}^{\rm (pert.)} m_{\rm OS} = \delta_{np}^{\rm (pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{QCD}!$$

On the other hand

$$M_{\Upsilon(1S)} = m_{\rm OS}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \cdots) + \mathcal{O}\left(\frac{\Lambda_{QCD}^3}{(m_{\rm OS}\alpha_s)^2}\right).$$

$M_{\Upsilon(1S)}$ **is renormalon free. Therefore, the perturbative series suffers from renormalon ambiguities:** $A_n \sim n!$

$$\delta_{np}^{\rm (pert.)} M_{\Upsilon(1S)} = \delta_{np}^{\rm (pert.)} m_{\rm OS}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \cdots) \sim \Lambda_{QCD}.$$

Physics. Computations to n-loops produce small scales: me^{-n}. **From the effective field theory point of view these scales should be in the effective field theory instead that in the matching coefficients.**

Proposal: to subtract the renormalon from the matching coefficients.

OS mass

$$m_{\mathrm{OS}} = m_{\overline{\mathrm{MS}}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}\,,$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\mathrm{OS}}](t(u)) = N_m \nu \frac{1}{(1-2u)^{1+b}}\left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (\text{analytic term}),$$

Next renormalon at $u = 1$.

$$r_n \stackrel{n\to\infty}{=} N_m\,\nu\,\left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1 + \frac{b}{(n+b)}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right).$$

$$b = \frac{\beta_1}{2\beta_0^2}\,, \qquad c_1 = \frac{1}{4\,b\beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right)\,, \qquad \cdots$$

Determination of N_m

$$\begin{aligned} D_m(u) &= \sum_{n=0}^{\infty} D_m^{(n)} u^n = (1-2u)^{1+b} B[m_{\mathrm{OS}}](t(u)) \\ &= N_m \nu \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (1-2u)^{1+b}(\text{analytic term})\,. \end{aligned}$$

$$N_m \nu = D_m(u = 1/2).$$

$$\begin{aligned} N_m &= 0.4244 + 0.1379 + 0.0127 = 0.5750 \quad (n_f = 3) \\ &= 0.4244 + 0.1275 + 0.0004 = 0.5523 \quad (n_f = 4) \\ &= 0.4244 + 0.1199 - 0.0208 = 0.5235 \quad (n_f = 5) \end{aligned}$$

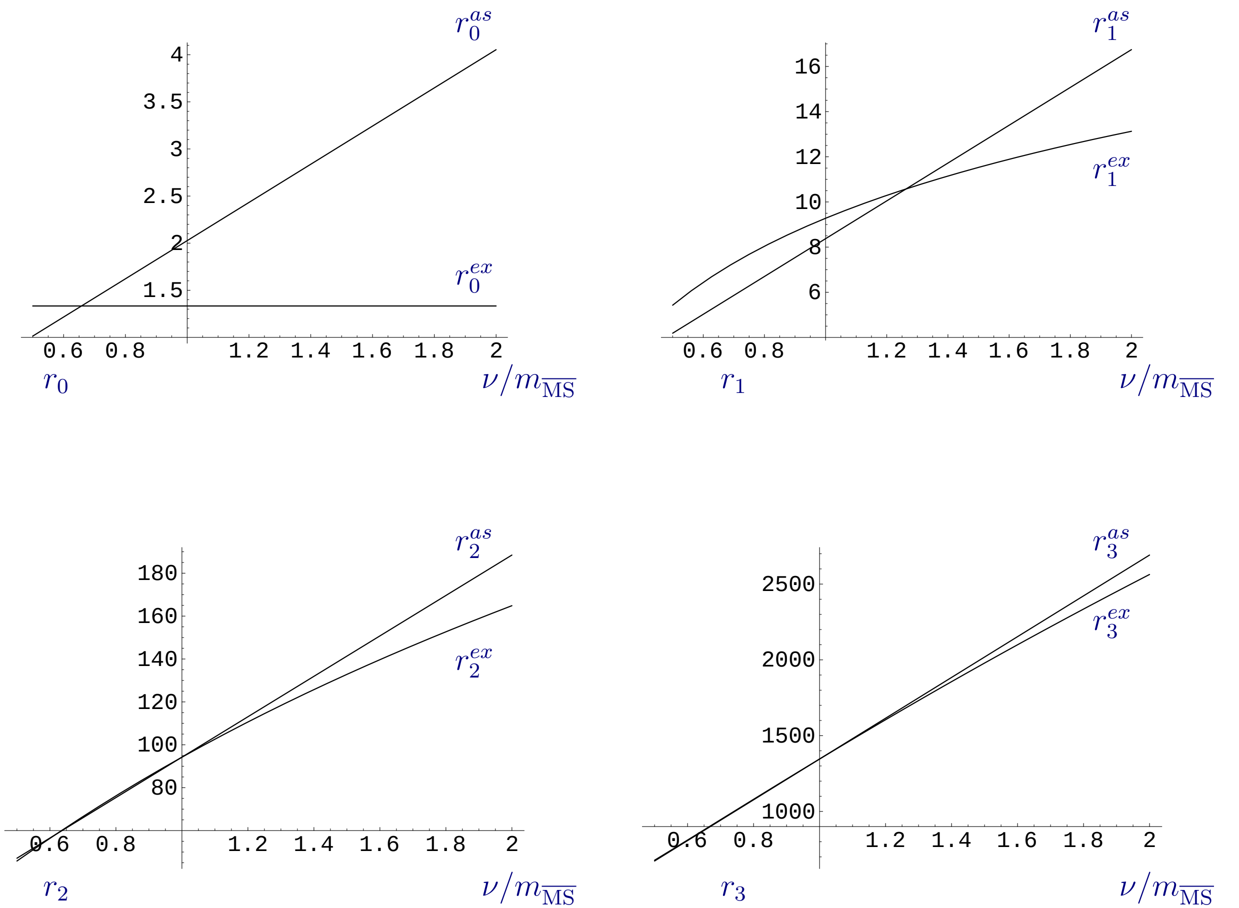


Figure 2: Plots of the exact (r_n^{ex}) and asymptotic (r_n^{as}) value of $r_n(\nu)$ at different orders in perturbation theory as a function of $\nu/m_{\overline{\rm MS}}$. The scale dependence of r_3^{ex} is known exactly. The constant term has been fixed using renormalon dominance.

$$r_n \overset{n\to\infty}{\sim} m_{\overline{\rm MS}} \left(\frac{\beta_0}{2\pi}\right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s[\nu/m_{\overline{\rm MS}}]}{s!},$$

$$r_n \overset{n\to\infty}{=} N_m \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right).$$

Renormalon subtracted matching and power counting

Effective field theory with renormalon free parameters but preserving the power counting rules.

The renormalon is associated to the non-analytic behavior in $1-2u$. These terms also exist in the effective theory. Procedure: to explicitly subtract them from the matching coefficients (the mass).

$$B[m_{\rm RS}] \equiv B[m_{\rm OS}] - N_m \nu_f \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right),$$

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \sum_{n=0}^{\infty} N_m \, \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)} \, .$$

Expansion in $\alpha_s(\nu)$

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS} \alpha_s^{n+1} \, ,$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}}, \nu, \nu_f)$. They are the ones expected to be of natural size. We now do not loose accuracy if we first obtain $m_{\rm RS}$ and later on $m_{\overline{\rm MS}}$.

Different scheme

$$B[m_{RS'}] \equiv B[m_{RS}] + N_m \nu_f \left(1 + c_1 + c_2 + \cdots\right) \, .$$

Check of convergence improvement

Masses	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
$m_{\rm OS}$	401	199	144	147	5 102
$m_{\rm RS}$	111	50	17	7	4 395
$m_{\rm RS'}$	401	114	38	15	4.778
$m_{\rm PS}$	210	80	42	— — —	4 542
$m_{1S}^{\rm (static)}$	102	50	19	8	4 389
$m_{\rm RS}$	256	95	40	21	4 622
$m_{\rm RS'}$	401	157	74	41	4.882
$m_{\rm PS}$	306	120	67	— — —	4.703
$m_{1S}^{\rm (static)}$	251	94	41	22	4 619

Table 1: *Contributions at various orders in α_s for different mass definitions for the bottom quark case, either with $\nu_f = 1/r = 2$ GeV (middle panel) or with $\nu_f = 1/r = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 4 has been used. The other parameters have been fixed to the values $m_{\overline{\rm MS}}(m_{\overline{\rm MS}}) = 4.21$ GeV, $\nu = m_{\overline{\rm MS}}(m_{\overline{\rm MS}})$ and $n_f = 4$.*

$$m_{1S}^{\rm (static)} \equiv m_{\rm OS} + \frac{V(r)}{2} = m_{\overline{\rm MS}} + \left(r_0 - \frac{C_f}{2r}\right)\alpha_s + \cdots .$$

HQET

$$\mathcal{L} = \bar{h}\,(iD_0 - \delta m_{\rm RS})\,h + O\left(\frac{1}{m_{RS}}\right) ,$$

where $\delta m_{\rm RS} = m_{\rm OS} - m_{\rm RS}$ and similarly for the NRQCD Lagrangian.

Weakly sensitive to long distance physics observable

$$\langle M_B \rangle - \langle M_D \rangle = m_{b,\rm RS} - m_{c,\rm RS} + \lambda_1\left(\frac{1}{2m_{b,\rm RS}} - \frac{1}{2m_{c,\rm RS}}\right) + O(1/m_{\rm RS}^2) .$$

pNRQCD. **If** $\Lambda_{QCD} \ll m\alpha_s$

$$V^{(0)}_{s,\mathrm{RS}(\mathrm{RS}')}(\nu_f) = V^{(0)}_s + 2\delta m_{\mathrm{RS}(\mathrm{RS}')} \,,$$

Check of convergence improvement

Potentials	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
$V_s^{(0)}$	-910	-306	-302	-383	$-1\,902$
$V_{s,\mathrm{RS}}^{(0)}$	-205	3	-2	-3	-208
$V_{s,\mathrm{RS}'}^{(0)}$	-910	-54	-14	-6	-984
$V_{s,\mathrm{PS}}^{(0)}$	-446	-42	-25	$---$	-513
$V_{s,\mathrm{RS}}^{(0)}$	-558	-63	-41	-26	-687
$V_{s,\mathrm{RS}'}^{(0)}$	-910	-180	-95	-54	$-1\,239$
$V_{s,\mathrm{PS}}^{(0)}$	-678	-116	-75	$---$	-869

Table 2: *Contributions at various orders in α_s for different singlet static potential definitions for some typical scales in the Υ system, either with $\nu_f = 2$ GeV (middle panel) or with $\nu_f = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 6 has been used. The other parameters have been fixed to the values $\nu = 1/r = 2.5$ GeV and $n_f = 4$.*

pNRQCD Lagrangian

$$\mathcal{L}^{(0)} = \mathrm{Tr}\Big\{S^\dagger\left(i\partial_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{s,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right) S + O^\dagger\left(iD_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{o,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right) O\Big\}$$
$$+\ gV_A(\mathbf{x})\mathrm{Tr}\left\{O^\dagger\mathbf{x}\cdot\mathbf{E}\,S + S^\dagger\mathbf{x}\cdot\mathbf{E}\,O\right\} + g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{O^\dagger\mathbf{x}\cdot\mathbf{E}\,O + O^\dagger O\mathbf{x}\cdot\mathbf{E}\right\},$$

Weakly sensitive to long distance physics observable

$$M_{nlj} = 2m_{\mathrm{RS}} + \sum_{m=2}^{\infty} A^{m,\mathrm{RS}}_{nlj}(\nu_{us})\alpha_s^m + \delta M^{\mathrm{US}}_{nlj}(\nu_{us})\,.$$

The static potential

$$V_s^{(0)}(r;\nu_{us}) = \sum_{n=0}^{\infty} V_{s,n}^{(0)}\alpha_s^{n+1},$$

$2m_{\rm OS} + V_s^{(0)}$ **(not** $2m_{\rm OS} + V_o^{(0)}$**) can be understood as an observable up to** $O(r^2\Lambda_{QCD}^3, \Lambda_{QCD}^2/m)$ **renormalon (and/or non-perturbative) contributions. We can use our knowledge of the asymptotic behavior of** $m_{\rm OS}$**.**

$$V_{s,n}^{(0)} \overset{n\to\infty}{=} N_V\,\nu\left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right)$$

$$2N_m + N_V = 0$$

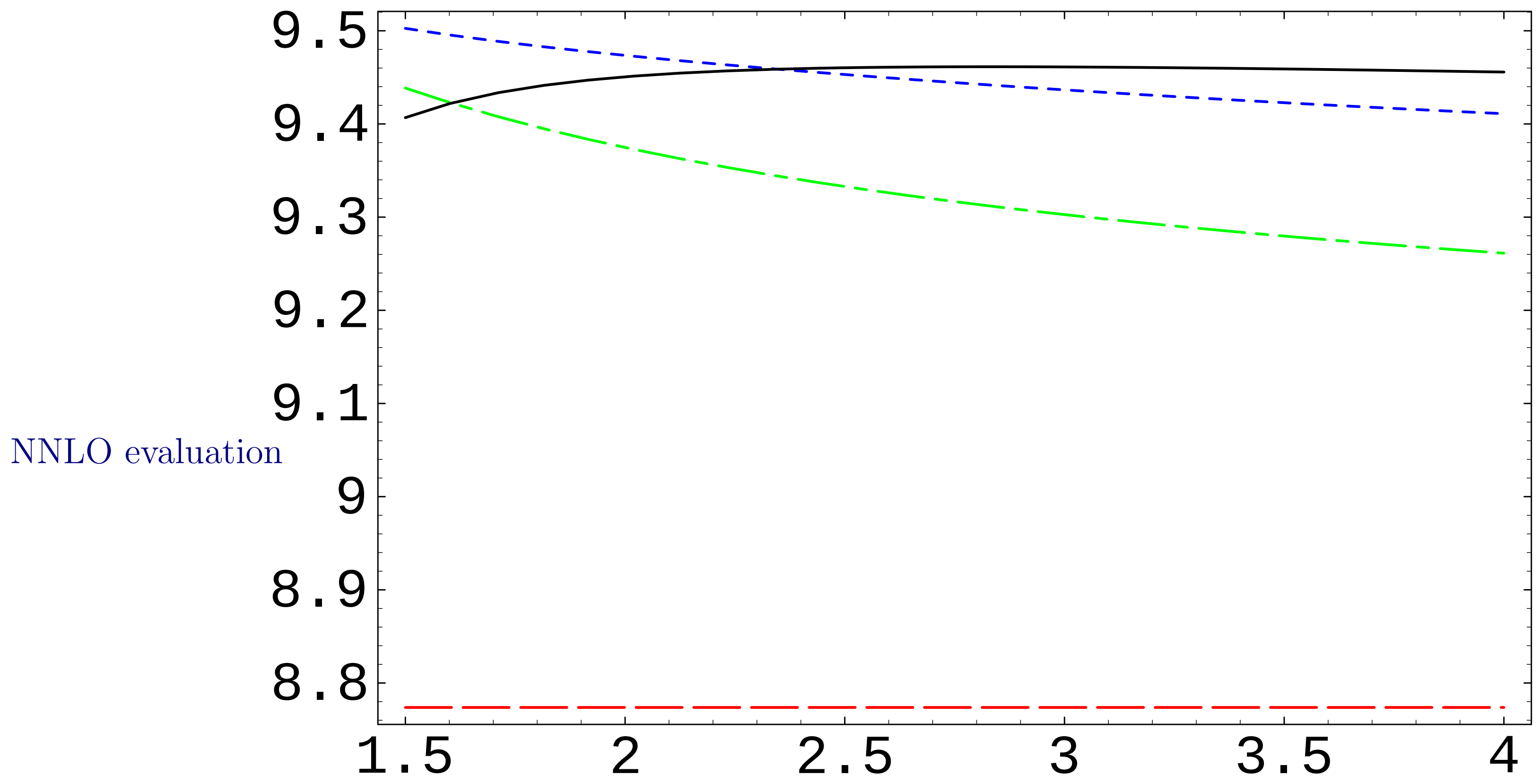
$$\begin{aligned} D_V(u) &= \sum_{n=0}^{\infty} D_V^{(n)}u^n = (1-2u)^{1+b}B[V_s^{(0)}](t(u)) \\ &= N_V\nu\left(1+c_1(1-2u)+c_2(1-2u)^2+\cdots\right)+(1-2u)^{1+b}(\text{analytic term}). \end{aligned}$$

Next (IR) renormalon at $u = 3/2$**.**

$$\begin{aligned} N_V &= -1.333+0.572-0.345 = -1.107 \quad (n_f=3) \\ &= -1.333+0.585-0.329 = -1.077 \quad (n_f=4) \\ &= -1.333+0.587-0.295 = -1.042 \quad (n_f=5)\,. \end{aligned}$$

$$2\frac{2N_m+N_V}{2N_m-N_V} = \begin{cases} 0.038 & ,\ n_f=3 \\ 0.025 & ,\ n_f=4 \\ 0.005 & ,\ n_f=5. \end{cases}$$

Bottom $\overline{\rm MS}$ quark mass determination



Dependence on the parameters for the RS scheme: $\nu = 2.5^{+1.5}_{-1}$ **GeV,** $\nu_f = 2 \pm 1$ **GeV,** $\alpha_s(M_z) = 0.118 \pm 0.003$ **and** $N_m = 0.552 \pm 0.0552$

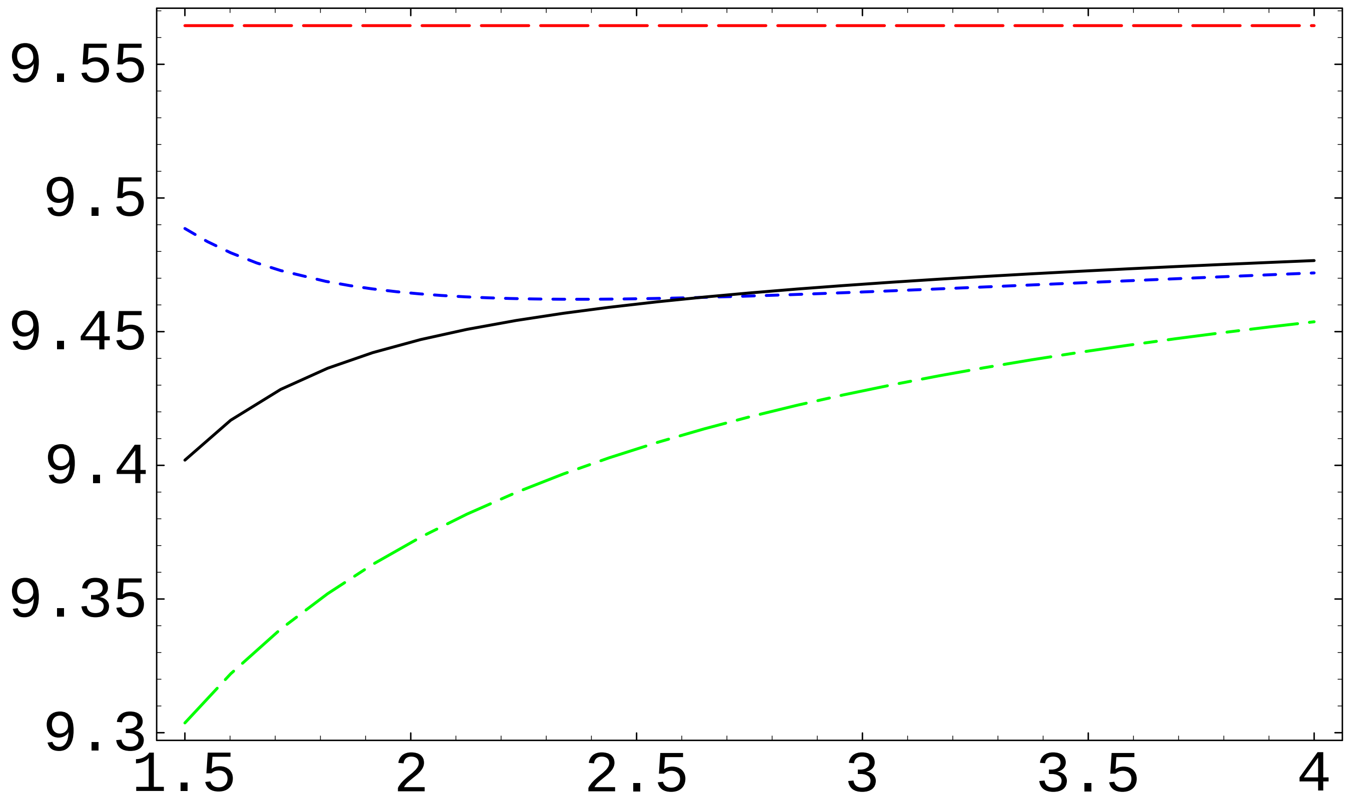
$$m_{b,\rm RS}(2\ {\rm GeV}) = 4\,387^{+2}_{+28}(\nu)^{-5}_{+7}(\nu_f)^{-16}_{+16}(\alpha_s)^{-68}_{+68}(N_m)\ {\rm MeV};$$

$$m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}}) = 4\,203^{+2}_{+25}(\nu)^{-5}_{+6}(\nu_f)^{-28}_{+27}(\alpha_s)^{-10}_{+10}(N_m)\ {\rm MeV}.$$

Convergence. In the RS scheme

$$M_{\Upsilon(1S)} = 8\,774 + 559 + 120 + 7\ {\rm MeV}.$$

NNLO(st. pot.) $\sim +62$ **MeV. NNLO(rel.)** ~ -55 **MeV.**



For the RS' scheme, we obtain the result

$$m_{b,\mathrm{RS}'}(2\ \mathrm{GeV}) = 4\,782^{-08}_{+31}(\nu)^{-7}_{+3}(\nu_f)^{+15}_{-12}(\alpha_s)^{-28}_{+28}(N_m)\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,214^{-08}_{+28}(\nu)^{-6}_{+3}(\nu_f)^{-25}_{+25}(\alpha_s)^{-9}_{+9}(N_m)\ \mathrm{MeV}.$$

Convergence. In the RS' scheme

$$M_{\Upsilon(1S)} = 9\,564 - 158 + 56 - 2\ \mathrm{MeV}\,.$$

NNLO(st. pot.) $\sim +45$ **MeV. NNLO(rel.)** ~ -47 **MeV.**

The static singlet potential

The introduction of renormalons allows to obtain agreement between lattice simulations and perturbation theory.

$$E_s = 2m_{\rm OS} + V_{s,{\rm OS}} + \mathcal{O}(r^2)$$

$$E_s = 2m_{\rm RS}(\nu_f) + V_{s,{\rm RS}}(\nu_f) + \mathcal{O}(r^2)$$

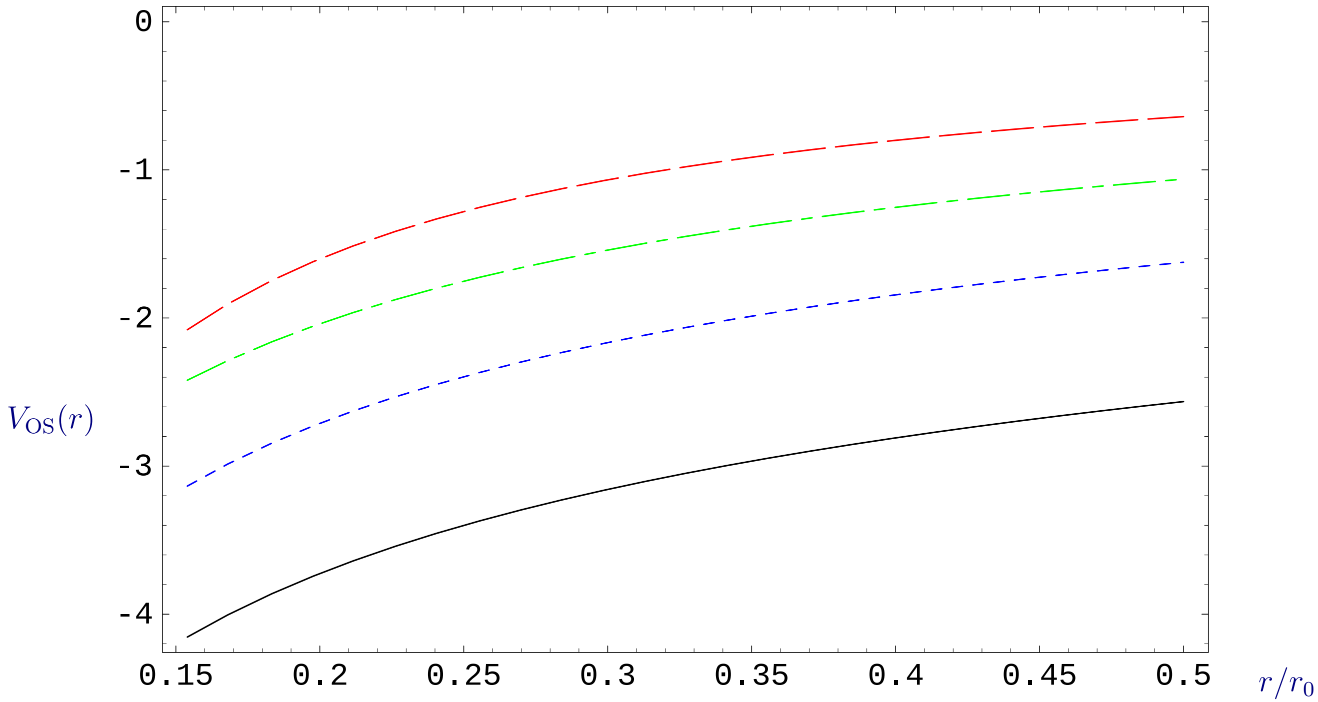


Figure 3: *Plot of $V_{\rm OS}(r)$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line). For the scale of $\alpha_s(\nu)$ $\nu =$ constant. $\nu_{us} = 2.5\, r_0^{-1}$.*

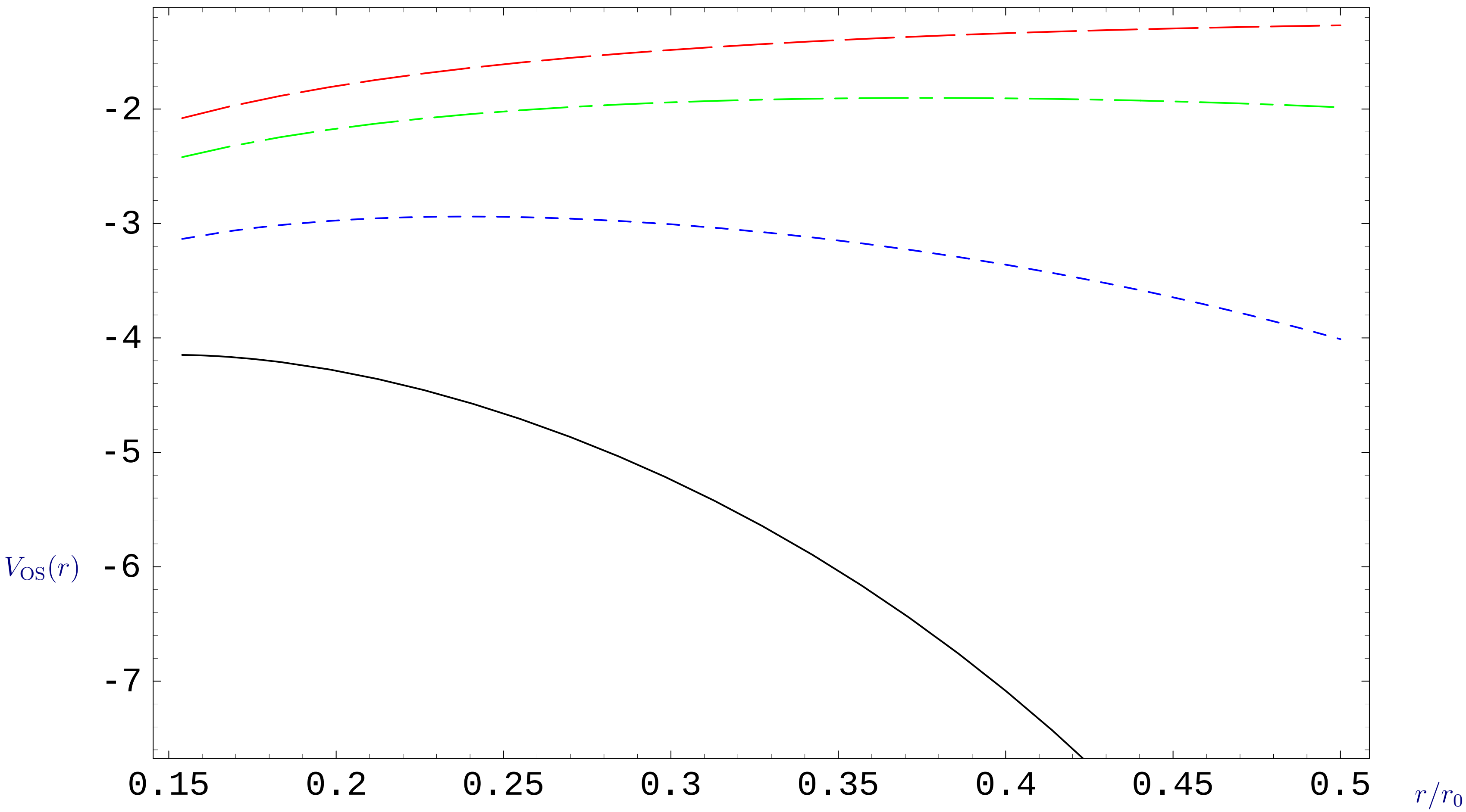


Figure 4: *Plot of* $V_{\rm OS}(r)$ *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line). For the scale of* $\alpha_s(\nu)$*, we set* $\nu = 1/r$*.* $\nu_{us} = 2.5\, r_0^{-1}$*.*

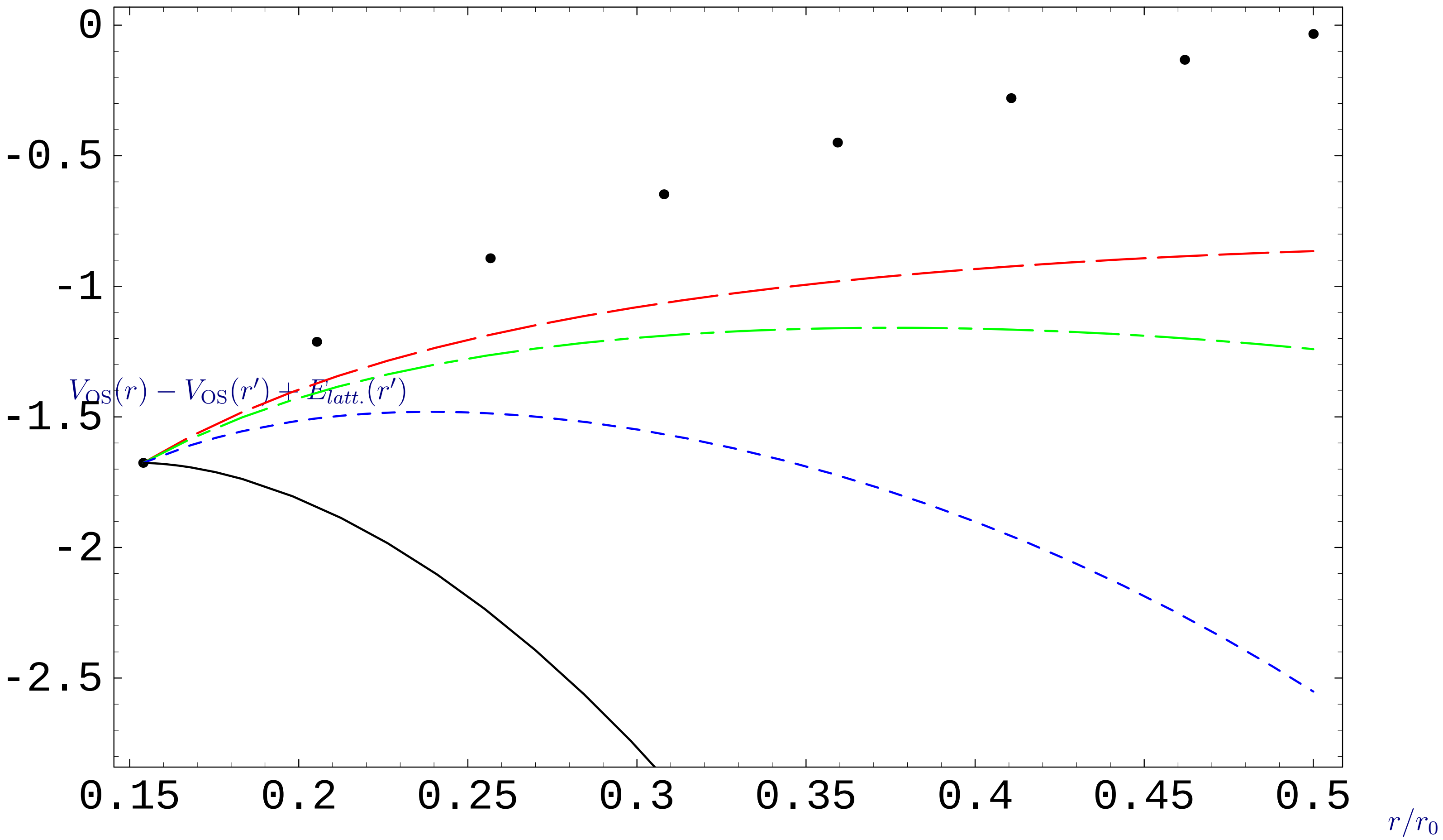


Figure 5: *Plot of $V_{\rm OS}(r) - V_{\rm OS}(r') + E_{latt.}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu = 1/r$. $\nu_{us} = 2.5\, r_0^{-1}$.*

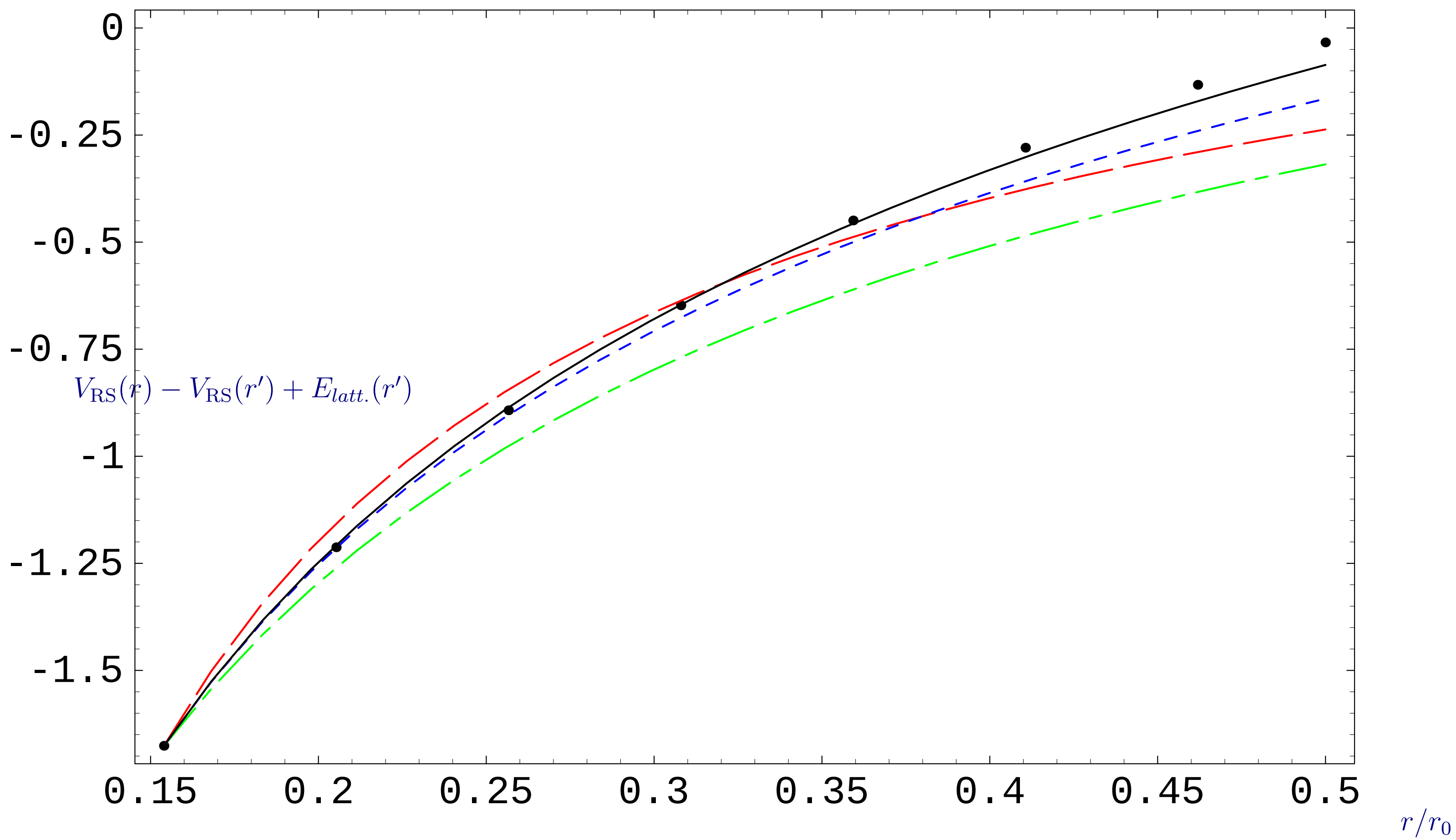


Figure 6: *Plot of $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$ $\nu =$ constant. $\nu_{us} = 2.5\, r_0^{-1}$.*

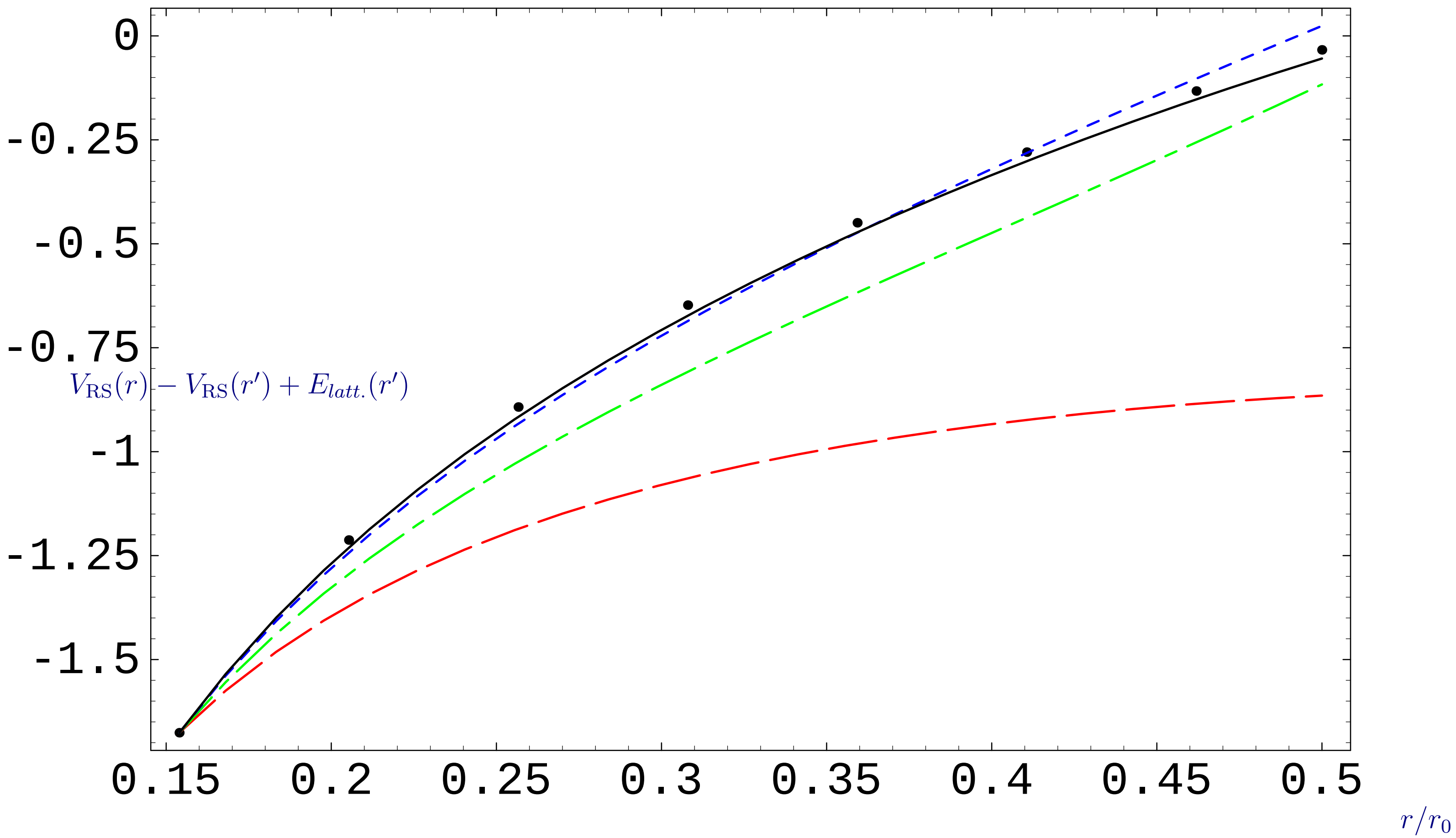


Figure 7: *Plot of* $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ *versus* r *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$*, we set* $\nu = 1/r$. $\nu_f = \nu_{us} = 2.5\, r_0^{-1}$.

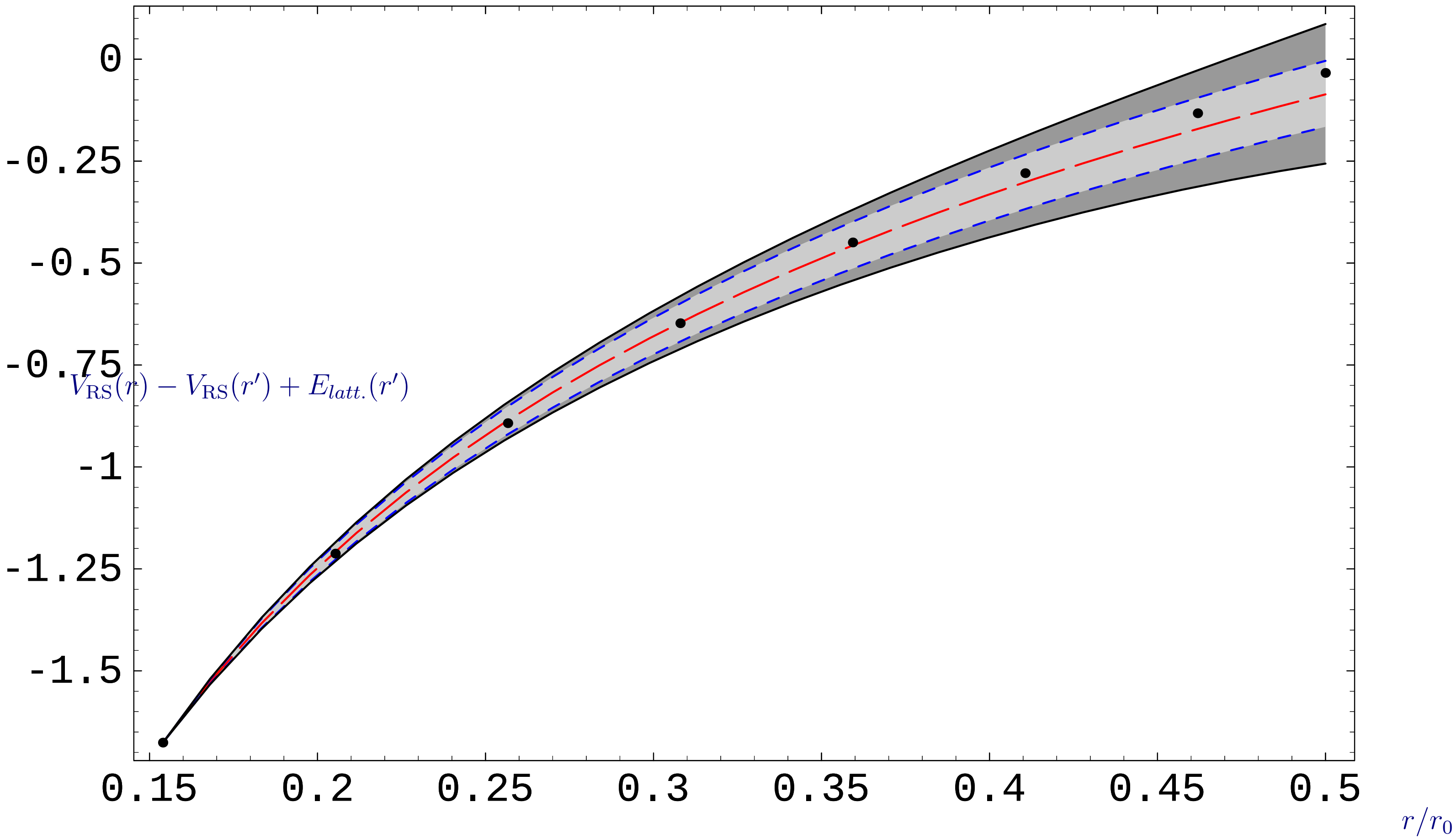


Figure 8: *Plot of* $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ *versus* r *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$ $\nu = constant$. $\nu_{us} = 2.5\, r_0^{-1}$.

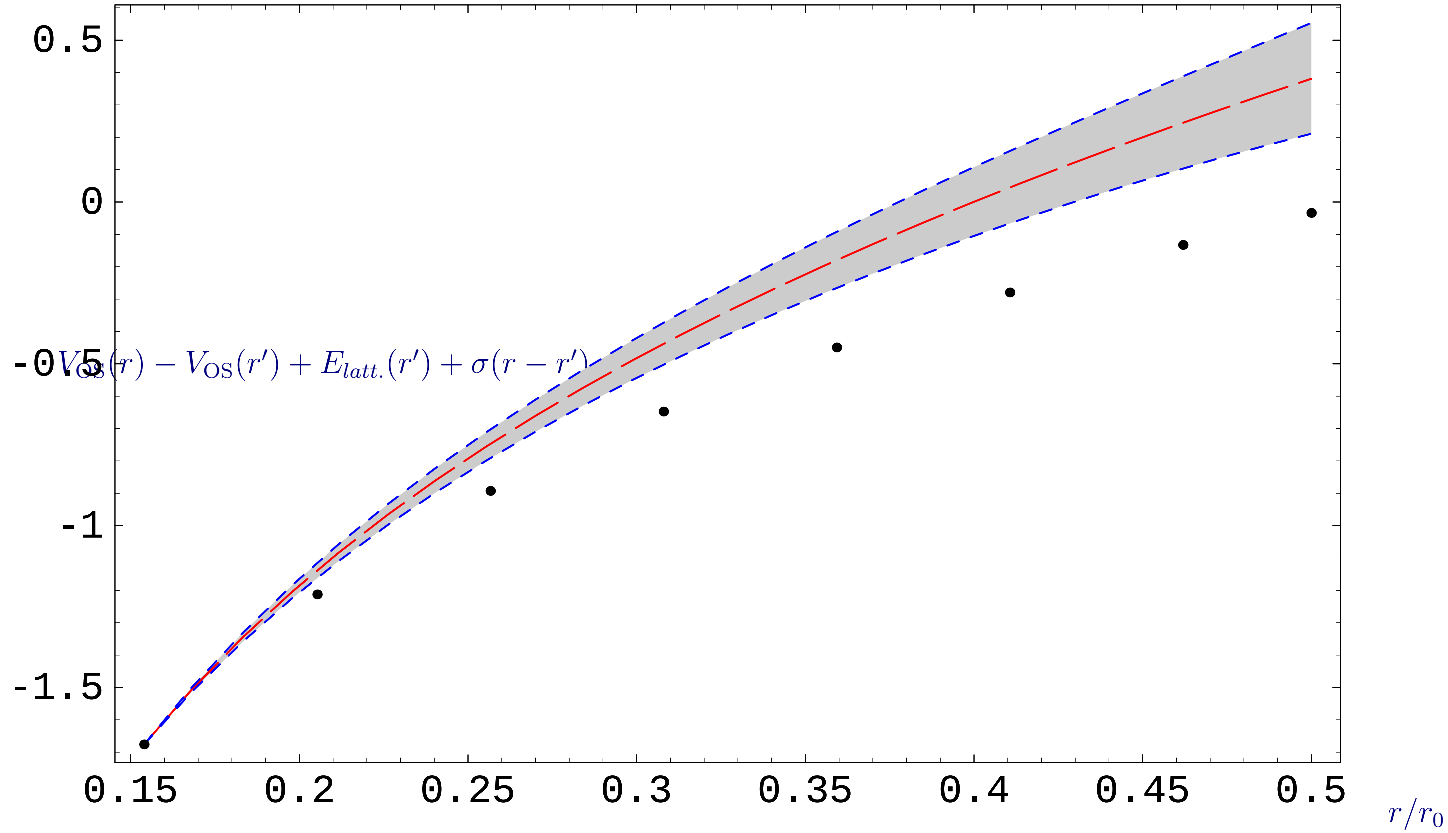


Figure 9: *Plot of* $V_{\rm OS}(r) - V_{\rm OS}(r') + E_{latt.}(r') + \sigma(r - r')$ *versus* r *at three loops (estimate) with the leading ultrasoft log compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$*, we set* $\nu =$ *constant.* $\sigma = 1.35\, r_0^{-2}$ *and* $\nu_{us} = 2.5\, r_0^{-1}$.

Constraint on the size of nonperturbative effecs for heavy quarkonium.
No linear non-perturbative potential at short distances.

OBSERVABLES

QED

Similar techniques. So far, most of the work done for purely leptonic systems. positronium, muonium...

Much better measurements can now be obtained using laser spectroscopy.

High precision measurements can provide tests of new physics. The paradigm is the decay of orthopositronium for which some experiments give values far of the theoretical ones. The theoretical precision is

$$\delta\Gamma/\Gamma^{(0)} \sim O(\alpha^3 \log \alpha)$$

These last results have been obtained using effective field theory methods.
$O(\alpha^2)$ Czarnecki, Melnikov, Yelkhovsky; Adkins, Fell, Sapirstein
$O(\alpha^3 \log \alpha)$ Hill, Lepage; Kniehl, Penin; Melnikov, Yelkhovsky.
($O(\alpha^3 \log^2 \alpha)$ known time ago. Karshenboim)

Some splittings of Positronium also seem to be problematic, for which, more or less, an equivalent level of precision has been achieved.

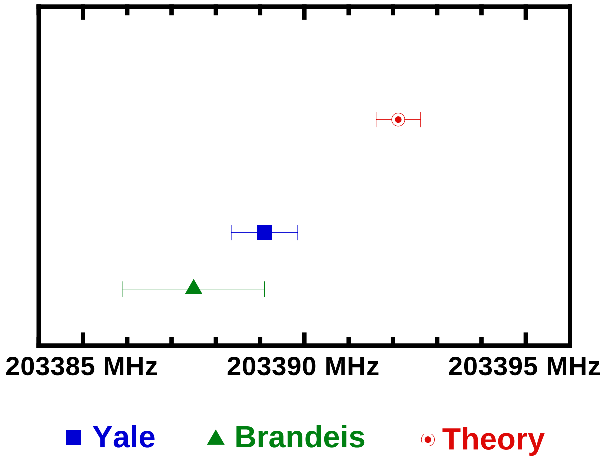


Figure 10: hyperfine splitting of 1S in positronium. From Karshenboim, hep-ph/0201241.

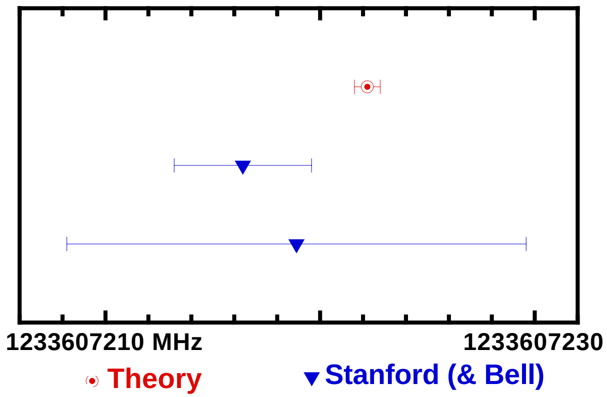


Figure 11: 1S-2S interval in positronium. From Karshenboim, hep-ph/0201241.

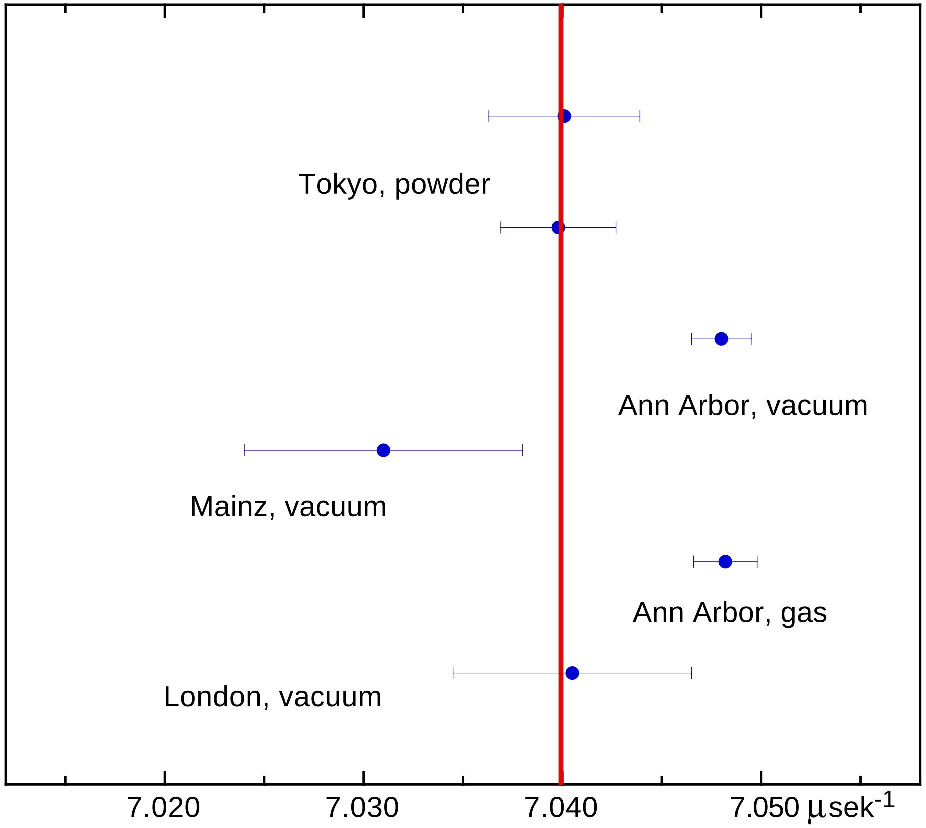


Figure 12: Decay of orthopositronium. From Karshenboim, hep-ph/0201241.

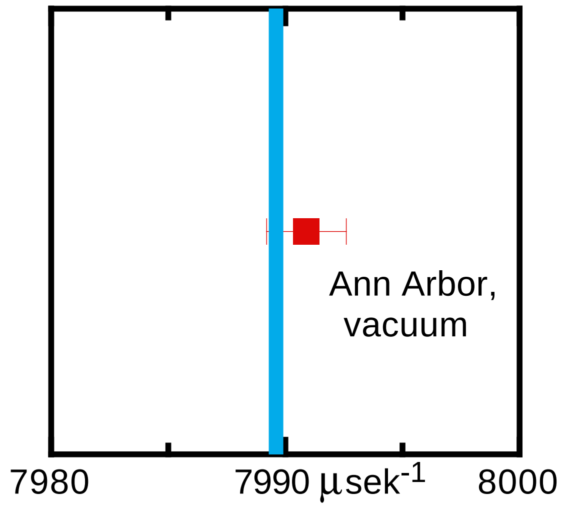


Figure 13: Decay of parapositronium. From Karshenboim, hep-ph/0201241.

HEAVY QUARKONIUM DECAYS

Pineda-Signer

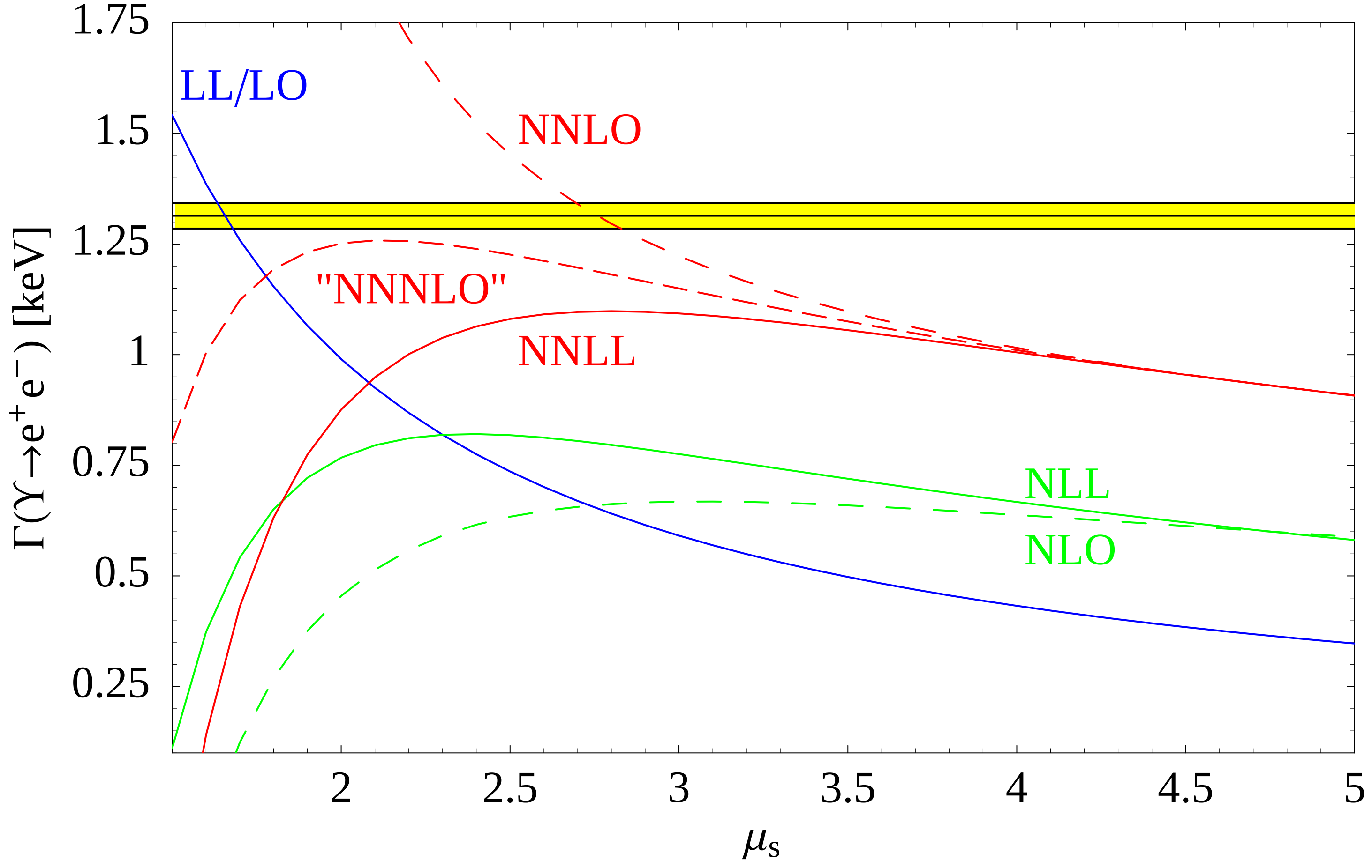


Figure 14: Prediction for the $\Upsilon(1S)$ decay rate to e^+e^-. We work in the RS' scheme.

The effect of the resummation of logarithms is important if compared with just keeping the single logarithm.

$$\Gamma(\Upsilon(nS) \to e^+e^-) = 16\pi \frac{C_A}{3} \left[\frac{\alpha_{EM}\, e_Q}{M_{\Upsilon(nS)}}\right]^2 \left|\phi_n^{(s=1)}(\mathbf{0})\right|^2 \left\{B_1 - d_1 \frac{M_{\Upsilon(nS)} - 2m_Q}{6m_Q}\right\}^2$$

$$\Gamma(\eta_b(nS) \to \gamma\gamma) = 16\pi\, C_A \left[\frac{\alpha_{EM}\, e_Q^2}{M_{\eta_b(nS)}}\right]^2 \left|\phi_n^{(s=0)}(\mathbf{0})\right|^2 \left\{B_0 - d_0 \frac{M_{\eta_b(nS)} - 2m_Q}{6m_Q}\right\}^2 .$$

The corrections to the wave function at the origin are obtained by taking the residue of the Green function at the position of the poles

$$\left|\phi_n^{(s)}(\mathbf{0})\right|^2 = \left|\phi_n^{(0)}(\mathbf{0})\right|^2 \left(1 + \delta\phi_n^{(s)}\right) = \underset{E=E_n}{\mathbf{Res}}\, G_s(\mathbf{0}, \mathbf{0}; E)\,,$$

where the LO wave function is given by

$$\left|\phi_n^{(0)}(\mathbf{0})\right|^2 = \frac{1}{\pi} \left(\frac{m_Q C_F \alpha_s}{2n}\right)^3 .$$

Note that $\left|\phi_n^{(s)}(\mathbf{0})\right|^2$ is SCHEME and SCALE dependent.

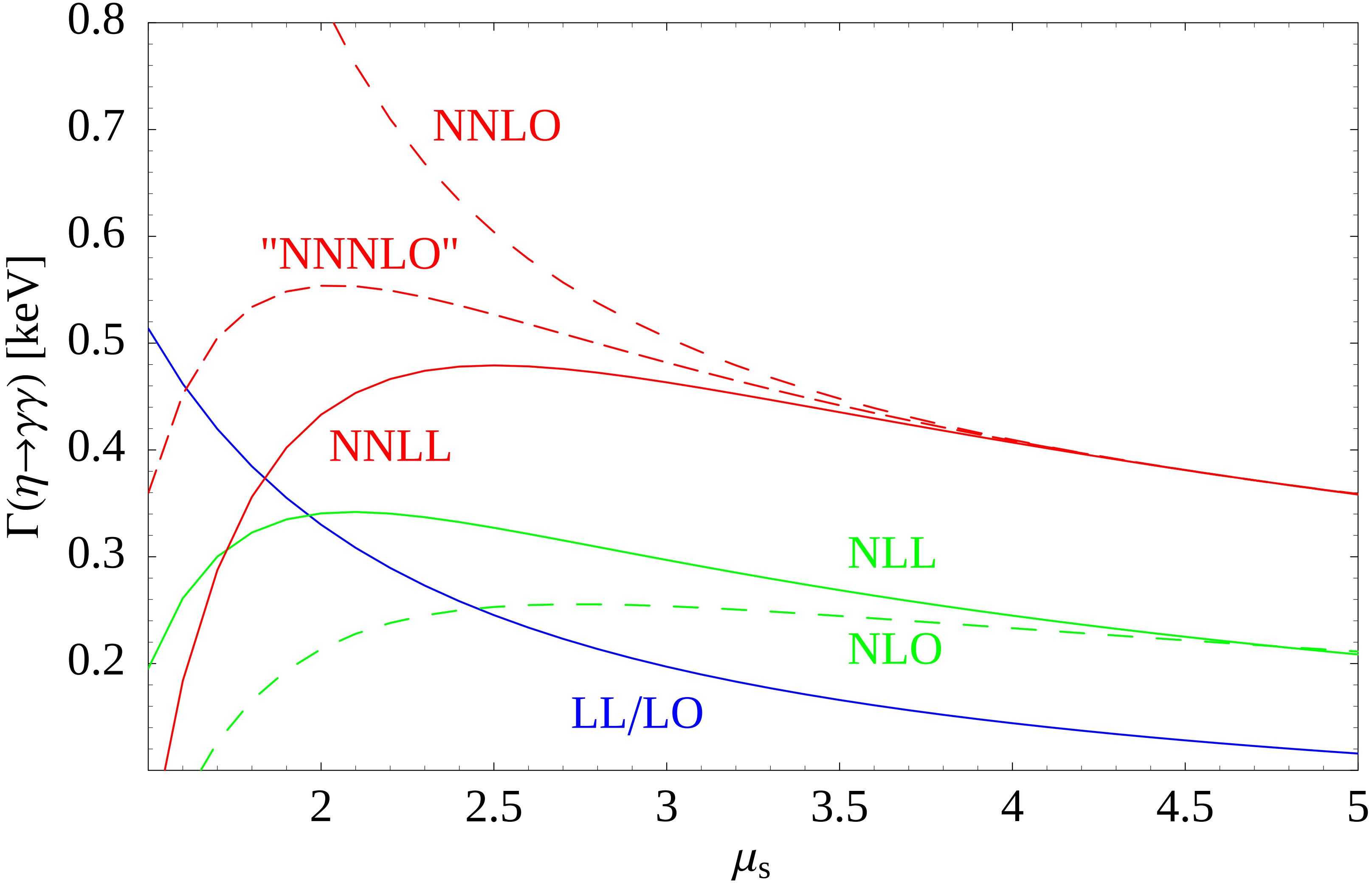


Figure 15: Prediction for the $\eta_b(1S)$ decay rate to two photons. We work in the RS' scheme.

NLL HYPERFINE SPLITTING

Kniehl, Penin, Smirnov, Steinhauser, Pineda;
Penin, Smirnov, Steinhauser, Pineda

$$\delta E \sim m\alpha^4 + m\alpha^5 \ln\alpha + m\alpha^6 \ln^2\alpha + \cdots$$
$$+ \; m\alpha^5 + m\alpha^6 \ln\alpha + m\alpha^7 \ln^2\alpha + m\alpha^8 \ln^3\alpha + \cdots$$

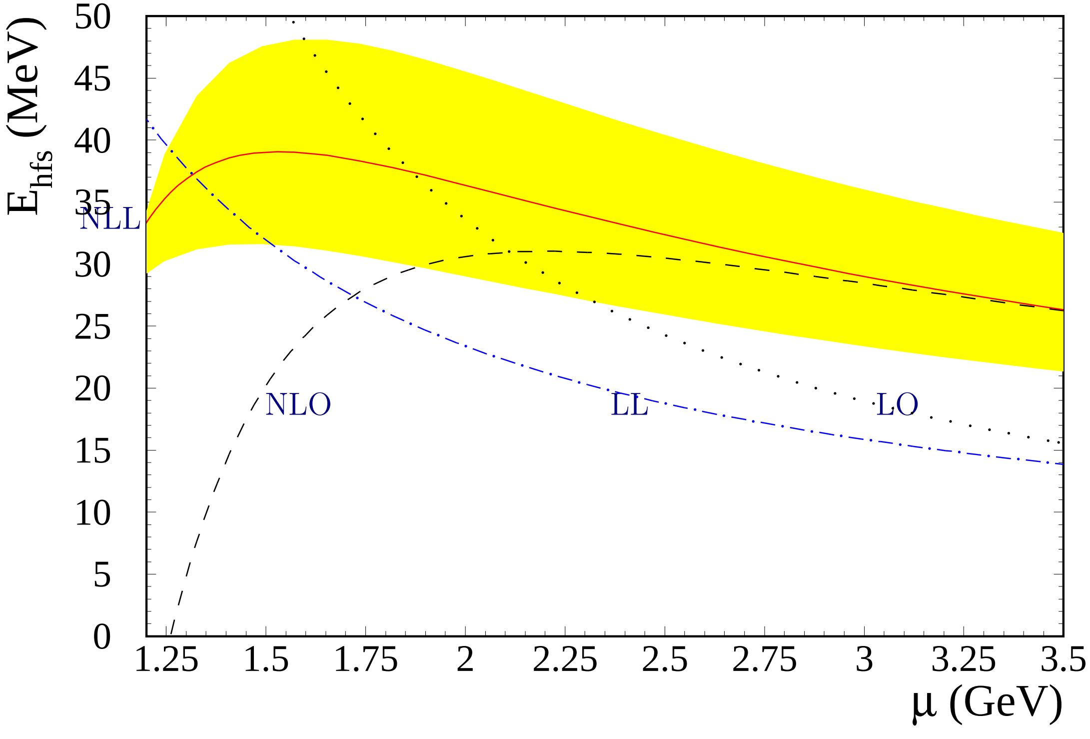


Figure 16: HFS for 1S bottomonium as the function of the renormalization scale μ in LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximation. For the NLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$.

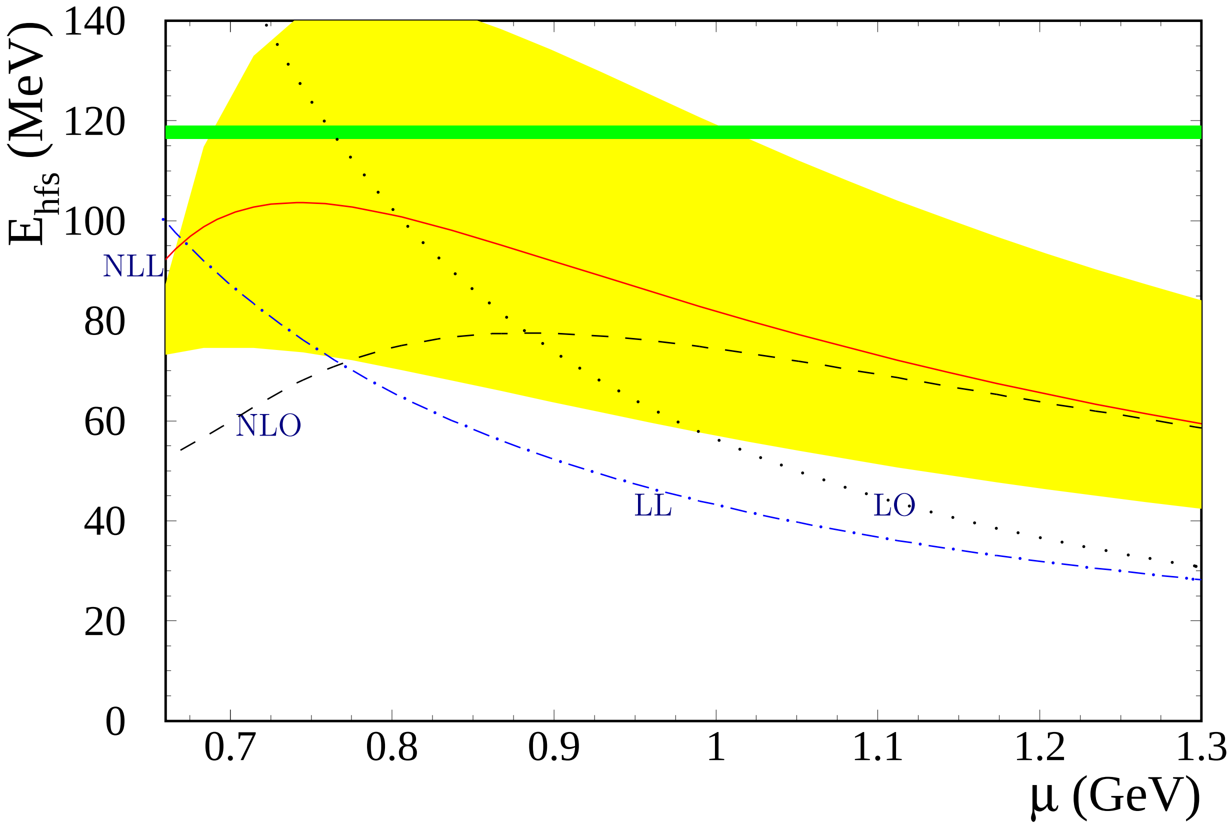


Figure 17: HFS for 1S charmonium as the function of the renormalization scale μ in LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximation versus the experimental result. For the NLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$. The horizontal band gives the experimental value and its error.

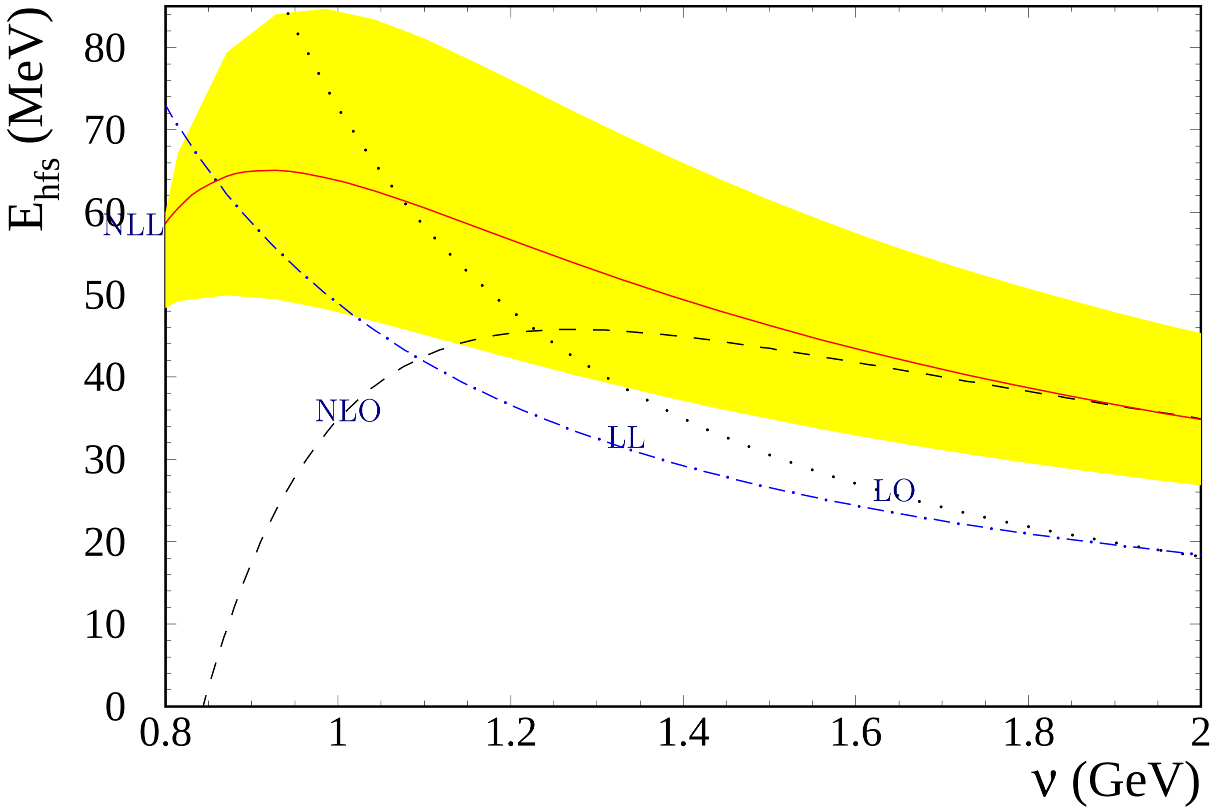


Figure 18: HFS for charm-bottom quarkonium as the function of the renormalization scale ν in LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximation for $\nu_h = 1.95$ GeV. For the NLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$.

$$M(\eta_b) = 9421 \pm 11\,(\mathrm{th})\,^{+9}_{-8}\,(\delta\alpha_s)\ \ \mathrm{MeV}$$
$$M(B_c^*) - M(B_c) = 65 \pm 24\,(\mathrm{th})\,^{+19}_{-16}\,(\delta\alpha_s)\ \ \mathbf{MeV}$$

NLL Hyperfine splitting; analytic result

$$E_{HF} = -\frac{4}{3}E_n\alpha_s\frac{C_f^2\delta_{l0}\delta_{s1}}{n}\left\{(1+2\delta\phi_n)\alpha_s(m)\left(1+\frac{2\beta_0-7C_A}{2\beta_0-4C_A}\left\{z^{-2C_A+\beta_0}-1\right\}\right)\right.$$
$$+\left(D^{(2)}_{S^2,s}(\nu_s=\nu_p=m)-\alpha_s(m)\right)$$
$$\left.+\left(\ln\left(\frac{mC_f\alpha_s}{\nu_p n}\right)+\sum_{k=1}^{n}\frac{1}{k}+\frac{n-1}{2n}\right)\alpha_s c_F^2\gamma_{D^{(2)}_{S^2,s}}+\delta D^{(2)}_{S^2,s}(\nu_s)|_{NLL}+\delta D^{(2)}_{S^2,s}(\nu_p)|_{NLL}\right\},$$

where $\Psi_n(z)=\frac{d^n\ln\Gamma(z)}{dz^n}$ **and** $\Gamma(z)$ **is the Euler** Γ**-function and**

$$\delta\phi_n=\frac{\alpha_s}{\pi}\left[-C_A+\frac{\beta_0}{4}\left(3\ln\left(\frac{\nu_p n}{mC_f\alpha_s}\right)+\Psi_1(n+1)-2n\Psi_2(n)+\frac{3}{2}+\gamma_E+\frac{2}{n}\right)\right].$$

$$D^{(2)}_{S^2,s}(\nu_s=\nu_p=m)=\alpha_s(m)\left\{1+\left[-\frac{5}{9}T_Fn_f+\frac{3}{2}(1-\ln 2)T_F+\frac{11C_A-9C_f}{18}\right]\frac{\alpha_s(m)}{\pi}\right\}$$

$$\delta D^{(2)}_{S^2,s}(\nu_s)|_{NLL}=B_1\alpha_s^2(m)(z^{-\gamma_0+\beta_0}-1)+B_2\alpha_s^2(m)(z^{-\gamma_0+2\beta_0}-1)$$

where

$$B_1=\frac{\beta_1\gamma_0-\beta_0\,(2\,\beta_0\,c_1+\gamma_1)}{2\,{\beta_0}^2\,(\beta_0-\gamma_0)\,\pi}\gamma^{(1)}_{D^{(2)}_{S^2,s}}$$

$$B_2=\frac{-\beta_1\,\gamma_0\,\gamma^{(1)}_{D^{(2)}_{S^2,s}}+\beta_0\,\gamma_1\,\gamma^{(1)}_{D^{(2)}_{S^2,s}}+\beta_0\left(\beta_1\,\gamma^{(1)}_{D^{(2)}_{S^2,s}}-4\,\beta_0\,\gamma^{(2)}_{D^{(2)}_{S^2,s}}\right)}{2\,{\beta_0}^2\,(2\,\beta_0-\gamma_0)\,\pi}$$

$$\delta D^{(2)}_{S^2,s}(\nu_p)|_{NLL} = A_1\alpha_s^2(m)\ln(z^{\beta_0}) + A_2\alpha_s^2(m)(z^{\beta_0}-1) + A_3\alpha_s^2(m)\left(z^{2\beta_0}-1\right)$$
$$+A_4\alpha_s^2(m)\left(z^{\frac{-13C_A}{6}+\beta_0}-1\right) + A_5\alpha_s^2(m)\left(z^{-2C_A+\beta_0}-1\right) + A_6\alpha_s^2(m)\left(z^{\frac{-25C_A}{6}+2\beta_0}-1\right)$$
$$+A_7\alpha_s^2(m)\left(z^{-4C_A+2\beta_0}-1\right) + A_8\alpha_s^2(m)\left(z^{-3C_A+2\beta_0}-1\right) + A_9\alpha_s^2(m)\left(z^{-2C_A+2\beta_0}-1\right)$$
$$+A_{10}\alpha_s^2(m)\left(z^{-C_A+2\beta_0}-1\right) + A_{11}\alpha_s^2(m)\left(z^{-\frac{2(C_A-2n_fT_F)}{3}+\beta_0}-1\right)$$
$$+A_{12}\alpha_s^2(m)\left(z^{-\frac{2(4C_A-2n_fT_F)}{3}+2\beta_0}-1\right) + A_{13}\alpha_s^2(m)\left(1-z^{\beta_0}-(2-z^{\beta_0})\ln\left(2-z^{\beta_0}\right)\right)$$
$$+A_{14}\alpha_s^2(m)\Bigg(-HypergeometricPFQ(\{1,1,-1+\frac{2\,C_A}{\beta_0}\},\{2,2\},\frac{1}{2})$$
$$-\left(-2+z^{\beta_0}\right)\,HypergeometricPFQ(\{1,1,-1+\frac{2\,C_A}{\beta_0}\},\{2,2\},1-\frac{z^{\beta_0}}{2})$$
$$+\frac{\beta_0\left(4-4^{\frac{C_A}{\beta_0}}\,z^{2\beta_0-2\,C_A}\right)\,\log(2-z^{\beta_0})}{-4\,\beta_0+4\,C_A}\Bigg)$$
$$+A_{15}\alpha_s^2(m)\left((1-z^{\beta_0})(5+z^{\beta_0})+2(-4+z^{2\beta_0})\ln\left(2-z^{\beta_0}\right)\right)\;,$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_Fn_f$**,** $z = \left[\frac{\alpha_s(\nu_p)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}}$ **and** $w = \left[\frac{\alpha_s(\nu_p^2/m)}{\alpha_s(\nu_p)}\right]^{\frac{1}{\beta_0}}$**. The coefficients** A_i

read

$$
\begin{aligned}
A_1 &= \frac{3\,C_A\,{C_f}^2\,\pi}{2\,{\beta_0}^2\,(6\,\beta_0-13\,C_A)\,(\beta_0-2\,C_A)^2\,(3\,\beta_0-2\,C_A+4\,n_f\,T_F)} \\
&\quad\times\Big(-\big((6\,\beta_0-13\,C_A)\,\big(2\,{C_A}^2\,(-9\,C_A+56\,C_f)-2\,\beta_0\,C_A\,(40\,C_f+C_A\,(-9+s(s+1)))\\
&\quad+3\,{\beta_0}^2\,(4\,C_f+C_A\,(-6+s(s+1)))\big)\big)+4\,n_f\,\big(4\,{C_A}^2\,(9\,C_A-74\,C_f)\\
&\quad-6\,{\beta_0}^2\,(4\,C_f+C_A\,(-6+s(s+1)))+\beta_0\,C_A\,(196\,C_f+C_A\,(-90+13\,s(s+1)))\big)\,T_F\Big)\,, \\
A_2 &= \frac{3\,C_f\,\pi}{52\,{\beta_0}^2\,(\beta_0-2\,C_A)\,(C_A-2\,n_f\,T_F)} \\
&\quad\times\Big(13\,C_A\,\big(C_A\,C_f\,(-9\,C_A+56\,C_f)+\beta_0\,(4\,{C_A}^2+5\,C_A\,C_f-16\,{C_f}^2)\big) \\
&\quad-4\,\big(2\,C_A\,C_f\,(-9\,C_A+74\,C_f)+\beta_0\,(26\,{C_A}^2+19\,C_A\,C_f-32\,{C_f}^2)\big)\,n_f\,T_F\Big)\,, \\
A_3 &= \frac{{C_f}^2\,\pi}{16\,\beta_0}\,, \\
A_4 &= \frac{288\,(6\,\beta_0-31\,C_A)\,{C_f}^2\,(5\,C_A+8\,C_f)\,n_f\,\pi\,T_F}{13\,(6\,\beta_0-13\,C_A)^2\,(\beta_0-2\,C_A)\,(9\,C_A+8\,n_f\,T_F)}\,, \\
A_5 &= \frac{{C_f}^2\,\pi}{8\,(\beta_0-2\,C_A)^3} \\
&\quad\times\Bigg(4\,\Bigg(\frac{(\beta_0-2\,C_A)}{\beta_0\,(6\,\beta_0-13\,C_A)} \\
&\qquad\times\big(3\,C_A\,(30\,{\beta_0}^2-109\,\beta_0\,C_A+42\,{C_A}^2)-4\,(6\,\beta_0-37\,C_A)\,(2\,\beta_0-7\,C_A)\,C_f \\
&\qquad-2\,(2\,\beta_0-7\,C_A)\,C_A\,s-2\,(2\,\beta_0-7\,C_A)\,C_A\,s^2\big)
\end{aligned}
$$

$$
+\frac{27\,(\beta_0-5\,C_A)\,{C_A}^2}{C_A+n_f\,T_F}+\frac{216\,(2\,\beta_0-7\,C_A)\,{C_A}^2\,((\beta_0-3\,C_A)\,C_A-8\,(\beta_0-2\,C_A)\,C_f)}{\beta_0\,(6\,\beta_0-13\,C_A)\,(3\,\beta_0-2\,C_A+4\,n_f\,T_F)}\Bigg)\,,
$$

$$
A_6 = \frac{96\,(6\,\beta_0-31\,C_A)\,(2\,\beta_0-7\,C_A)\,{C_f}^2\,(5\,C_A+8\,C_f)\,n_f\,\pi\,T_F}{13\,(12\,\beta_0-25\,C_A)\,(6\,\beta_0-13\,C_A)\,(\beta_0-2\,C_A)\,C_A\,(9\,C_A+8\,n_f\,T_F)}\,,
$$

$$
\begin{aligned}
A_7 =\; & \frac{{C_f}^2\,\pi}{32\,(\beta_0-2\,C_A)^3\,(C_A+n_f\,T_F)} \\
& \times\Big(-2\,{C_A}^3\,(-371+62\,s\,(1+s))-4\,{C_A}^2\,n_f\,(-28+31\,s\,(1+s))\,T_F \\
& -{\beta_0}^2\,(C_A\,(-52+9\,s\,(1+s))+n_f\,(-16+9\,s\,(1+s))\,T_F) \\
& +2\,\beta_0\,C_A\,(C_A\,(-197+34\,s\,(1+s))+2\,n_f\,(-22+17\,s\,(1+s))\,T_F)\Big)\,,
\end{aligned}
$$

$$
A_8 = \frac{C_A\,C_f\,\pi}{-8\,\beta_0+12\,C_A}\,,
$$

$$
\begin{aligned}
A_9 =\; & \frac{C_f\,\pi}{104\,\beta_0\,(\beta_0-2\,C_A)\,(\beta_0-C_A)\,C_A\,(C_A-2\,n_f\,T_F)} \\
& \times\Big(13\,C_A\,\Big(4\,\beta_0\,(4\,\beta_0-11\,C_A)\,{C_A}^2+C_A\,(-10\,{\beta_0}^2-13\,\beta_0\,C_A+63\,{C_A}^2)\,C_f \\
& \qquad -8\,(2\,\beta_0-7\,C_A)^2\,{C_f}^2\Big)+4\,\Big(-26\,\beta_0\,(4\,\beta_0-11\,C_A)\,{C_A}^2 \\
& \qquad\qquad +C_A\,(92\,{\beta_0}^2-91\,\beta_0\,C_A-126\,{C_A}^2)\,C_f \\
& \qquad\qquad +4\,(8\,\beta_0-37\,C_A)\,(2\,\beta_0-7\,C_A)\,{C_f}^2\Big)\,n_f\,T_F\Big)\,,
\end{aligned}
$$

$$
A_{10} = -\frac{(C_A-3\,C_f)\,C_f\,\pi}{2\,\beta_0-C_A}\,,
$$

$$A_{11} = \frac{81\, {C_A}^2\, {C_f}^2\, \pi\, \left(-3\, \beta_0 + 11\, C_A - 4\, n_f\, T_F\right)\, \left(C_A\, \left(7\, C_A - 32\, C_f\right) + 4\, \left(C_A - 8\, C_f\right)\, n_f\, T_F\right)}{8\, \left(\beta_0 - 2\, C_A\right)\, \left(C_A - 2\, n_f\, T_F\right)\, \left(C_A + n_f\, T_F\right)\, \left(3\, \beta_0 - 2\, C_A + 4\, n_f\, T_F\right)^2\, \left(9\, C_A + 8\, n_f\, T_F\right)},$$

$$A_{12} = \frac{-27\, \left(2\, \beta_0 - 7\, C_A\right)\, C_A\, {C_f}^2\, \pi\, \left(3\, \beta_0 - 11\, C_A + 4\, n_f\, T_F\right)}{16\, \left(\beta_0 - 2\, C_A\right)\, \left(C_A - 2\, n_f\, T_F\right)\, \left(C_A + n_f\, T_F\right)\, \left(3\, \beta_0 - 4\, C_A + 2\, n_f\, T_F\right)} \times \frac{\left(C_A\, \left(7\, C_A - 32\, C_f\right) + 4\, \left(C_A - 8\, C_f\right)\, n_f\, T_F\right)}{\left(3\, \beta_0 - 2\, C_A + 4\, n_f\, T_F\right)\, \left(9\, C_A + 8\, n_f\, T_F\right)}$$

$$A_{13} = \frac{8\, C_f\, C_A\, (C_A^2 + 3 C_A C_f + 2 C_f^2) \pi}{\beta_0^2\, (\beta_0 - 2\, C_A)}$$

$$A_{14} = \frac{8\, C_f\, (C_f + C_A)\, (C_A + 2 C_f)(2\beta_0 - 7 C_A) 2^{1 - \frac{2 C_A}{\beta_0}}\, \pi}{3\, \beta_0^2\, (\beta_0 - 2\, C_A)},$$

$$A_{15} = \frac{C_A^3 \pi}{4 \beta_0^2}\, .$$

DECAY RATIO AT NNLL

Penin, Smirnov, Steinhauser, Pineda

$$\frac{\Gamma(V_Q(nS) \to e^+e^-)}{\Gamma(P_Q(nS) \to \gamma\gamma)} \sim 1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \cdots$$
$$+\alpha + \alpha^2 \ln \alpha + \alpha^3 \ln^2 \alpha + \cdots$$
$$+\alpha^2 + \alpha^3 \ln \alpha + \alpha^4 \ln^2 \alpha + \cdots$$

$$\frac{\Gamma(T(1S) \to e^+e^-)}{\Gamma(\eta_t(1S) \to \gamma\gamma)} = \frac{1}{3Q_t^2}(1 - 0.13198 - 0.0179492) \ .$$
$$\frac{\Gamma(\Upsilon(1S) \to e^+e^-)}{\Gamma(\eta_b(1S) \to \gamma\gamma)} = \frac{1}{3Q_b^2}(1 - 0.302 - 0.111) \ .$$
$$\frac{\Gamma(J/\Psi(1S) \to e^+e^-)}{\Gamma(\eta_c(1S) \to \gamma\gamma)} = \frac{1}{3Q_c^2}(1 - 0.51313 - 0.325764) \ .$$

$$\Gamma(\eta_b(1S) \to \gamma\gamma) = 0.659 \pm 0.089(\mathrm{th.})^{+0.019}_{-0.018}(\delta\alpha_\mathrm{s}) \pm 0.015(\mathrm{exp.}) \ \mathrm{KeV} \ ,$$

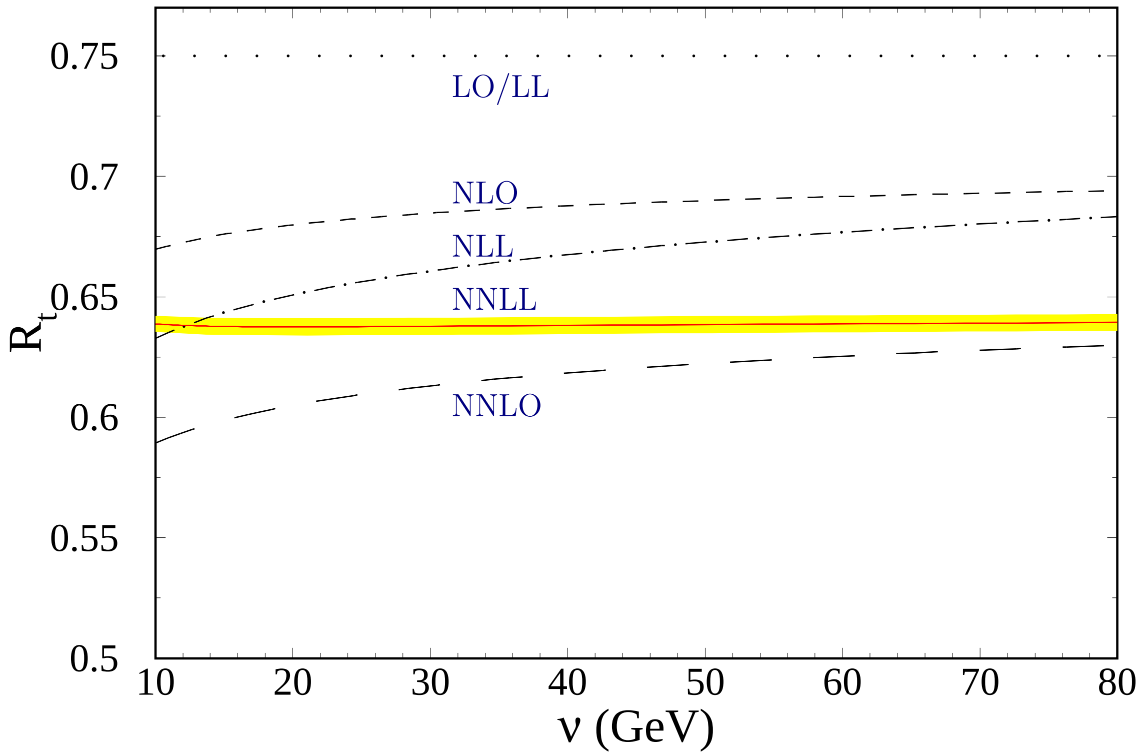


Figure 19: The spin ratio as the function of the renormalization scale ν in LO (dotted line), NLO (short-dashed line), NNLO (long-dashed line), LL (dotted line), NLL (dot-dashed line), and NNLL (bold solid line) approximation for the (would be) toponium ground state with $\nu_h = m_t$. For the NNLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$

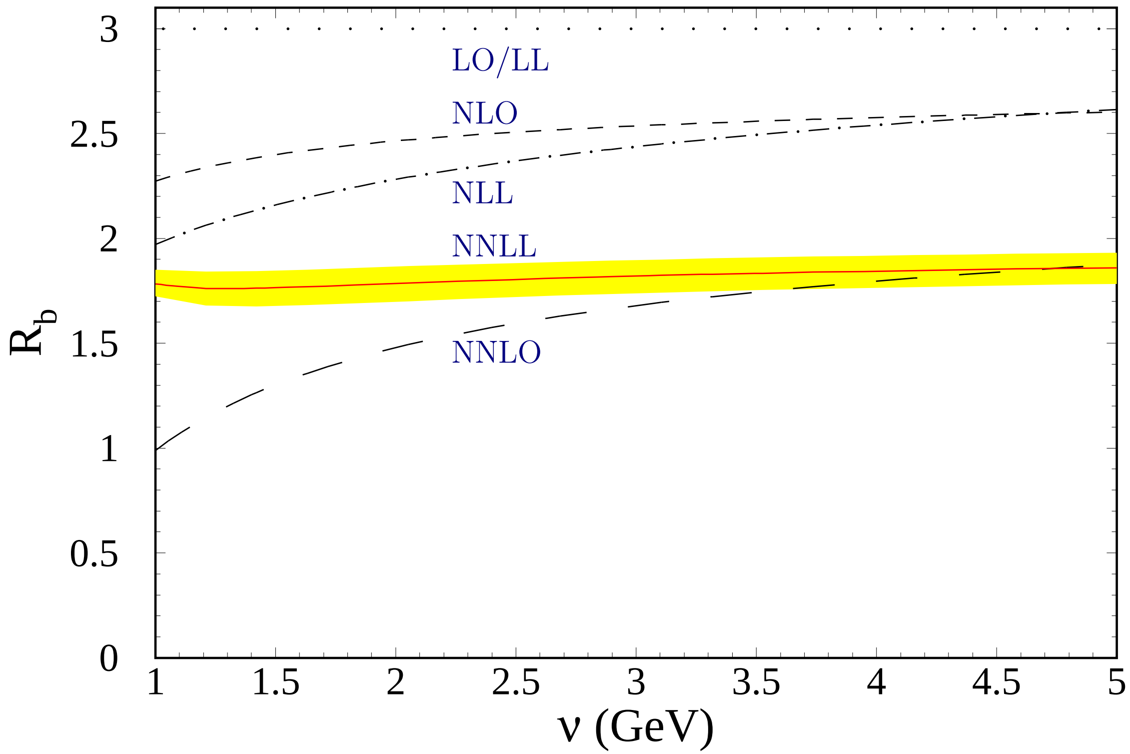


Figure 20: The spin ratio as the function of the renormalization scale ν in LO (dotted line), NLO (short-dashed line), NNLO (long-dashed line), LL (dotted line), NLL (dot-dashed line), and NNLL (bold solid line) approximation for the bottomonium ground state with $\nu_h = m_b$. For the NNLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$

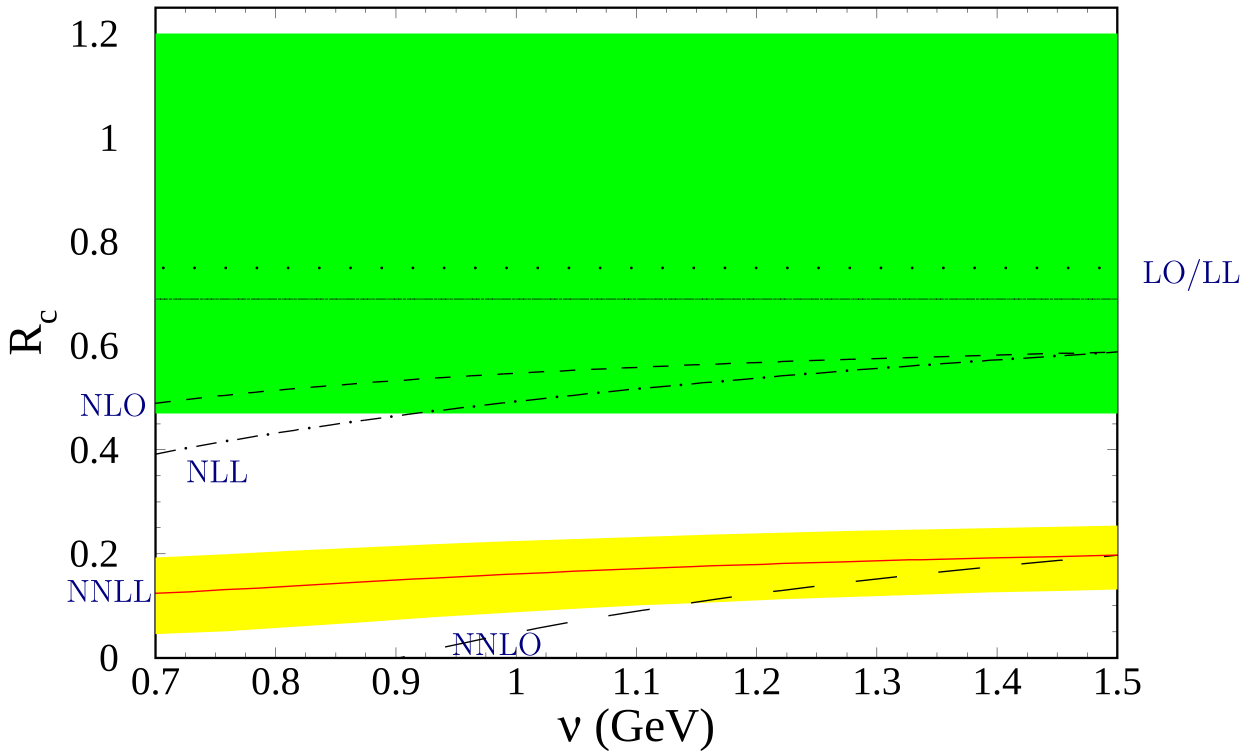


Figure 21: The spin ratio as the function of the renormalization scale ν in LO (dotted line), NLO (short-dashed line), NNLO (long-dashed line), LL (dotted line), NLL (dot-dashed line), and NNLL (bold solid line) approximation for the charmonium ground state with $\nu_h = m_c$. For the NNLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$. The horizontal band represents the experimental error of the ratio.

$$\frac{\Gamma(oPs \to 3\gamma)}{\Gamma(pPs \to 2\gamma)} = \frac{4(\pi^2-9)}{9\pi}\alpha\left\{1+\left(5-\frac{\pi^2}{4}+A_o\right)\frac{\alpha}{\pi}+\frac{7}{3}\alpha^2\ln\alpha\right.$$
$$+\left[\left(5-\frac{\pi^2}{4}\right)^2+\left(5-\frac{\pi^2}{4}\right)A_o+B_o-B_p\right]\left(\frac{\alpha}{\pi}\right)^2$$
$$-\left[-\frac{73}{9}-\frac{7\,A_o}{3}+\frac{7\,\pi^2}{12}+2\,\log(2)\right]\frac{\alpha^3}{\pi}\ln\alpha$$
$$\left.+\frac{83}{36}\alpha^4\ln^2\alpha-\frac{7}{6\pi}\alpha^5\ln^3\alpha+\cdots\right\},$$

where $A_o = 10.286606(10)$**,** $B_o = 44.87(26)$ **and** $B_p = 5.1243(33)$**.**

NON-RELATIVISTIC SUM RULES: BOTTOMONIUM

Pineda-Signer

$$M_n \equiv \frac{12\pi^2 e_b^2}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\big|_{q^2=0} = \int_0^\infty \frac{ds}{s^{n+1}} R_{b\bar{b}}(s),$$

$$M_n = 48\pi e_b^2 N_c \int_{-\infty}^{\infty} \frac{dE}{(E+2m_b)^{2n+3}}\left(B_1^2 - B_1 d_1 \frac{E}{3m_b}\right) \mathrm{Im}\, G(0,0,E)$$

n	$m_{b,\mathrm{PS}}$(2 GeV)	Δ_{th}	Δ_{exp}	Δ_α	Δ_{tot}	$\overline{m}_b$
6	4460	40	50	35	70	4135 ± 65
8	4505	45	25	30	60	4170 ± 55
10	4515	45	15	25	55	4185 ± 50
12	4520	45	10	20	50	4185 ± 45
14	4520	40	10	15	45	4185 ± 40
n	$m_{b,\mathrm{RS}}$(2 GeV)	Δ_{th}	Δ_{exp}	Δ_α	Δ_{tot}	$\overline{m}_b$
6	4315	55	50	25	80	4140 ± 70
8	4360	65	30	20	75	4180 ± 65
10	4370	65	20	10	70	4190 ± 60
12	4370	65	15	5	65	4190 ± 60
14	4370	65	10	5	65	4185 ± 55

Table 3: *Extraction of $m_{b,\mathrm{PS/RS}}$(2 GeV) with errors for various n. All values are given in MeV and rounded to 5 MeV. The total error has been obtained by adding the partial errors in quadrature. The corresponding value for the $\overline{\mathrm{MS}}$ mass with its error is given in the last column.*

small $n \to$ larger experimental error (not to use theoretical ansatz above threshold for experiment)
Large $n \to$ larger theoretical error, bad convergence of the perturbative series (it also depends on the scheme).

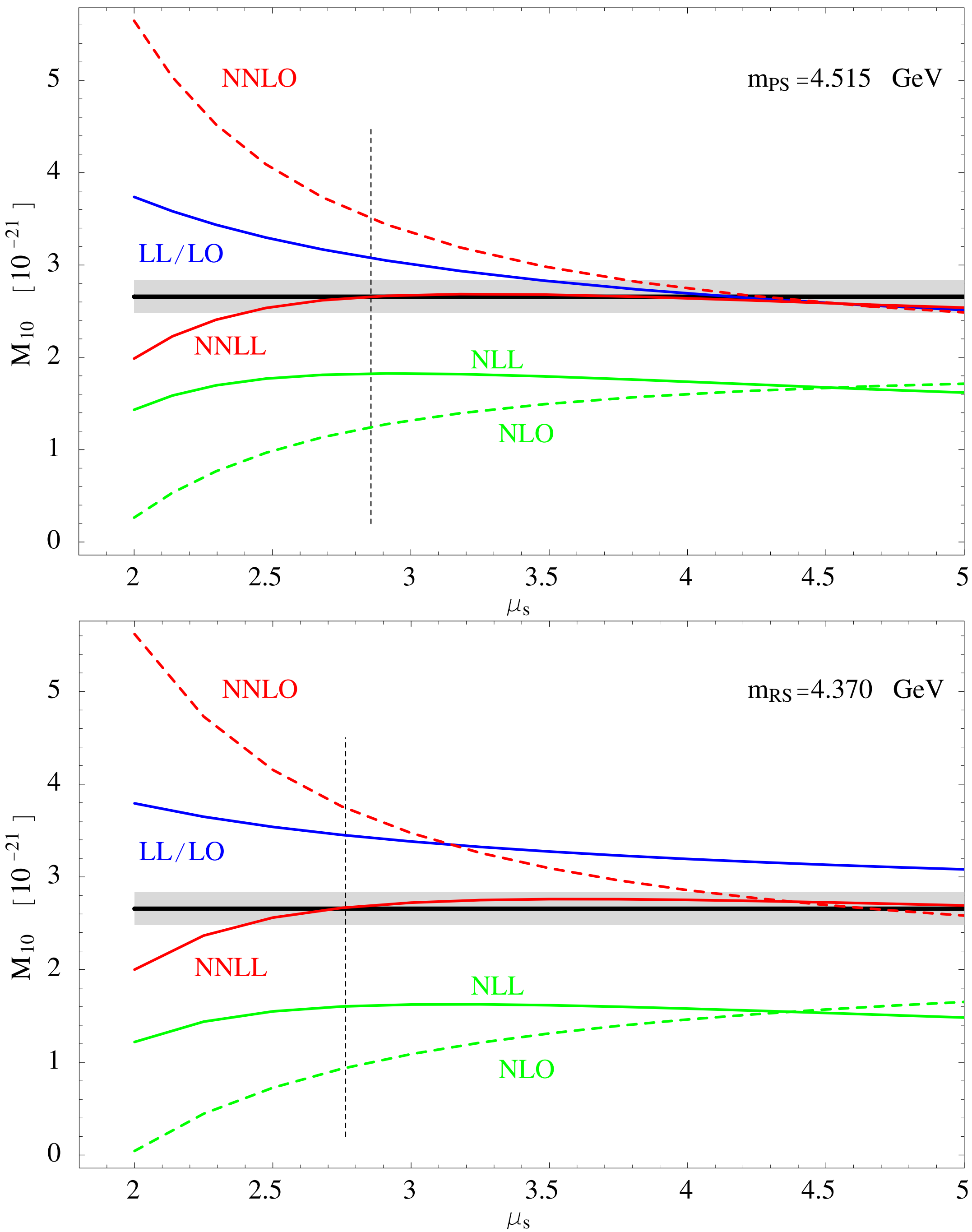


Figure 22: *The moment M_{10} as a function of μ_s at LO/LL, NLO, NLL, NNLO and NNLL for $m_{b\mathrm{PS}}$(2 GeV) = 4.515 GeV in the PS scheme (upper figure), and for $m_{b\mathrm{RS}}$(2 GeV) = 4.370 GeV in the RS scheme (lower figure). The experimental moment with its error is also shown (grey band).*

$$m_{b,\mathrm{PS}}(2\mathrm{GeV}) = 4.52 \pm 0.06 \text{ GeV},$$
$$m_{b,\mathrm{RS}}(2\mathrm{GeV}) = 4.37 \pm 0.07 \text{ GeV}.$$

$$\overline{m}_b(\overline{m}_b) = 4.19 \pm 0.06\,\mathrm{GeV}.$$

The perturbative series is sign-alternating. This is the opposite than for electromagnetic decays. The convergence of the perturbative series in sum rules is also better in sum rules than for electromagnetic decays.

NNLO determinations of the bottom sum rules suffer from very huge theoretical uncertainties (which are not always incorporated in the errors): bad scale dependence and bad convergence of the perturbative series. Therefore, they can not provide precise determinations of the bottom mass.

t–$\bar{t}$ NEAR THRESHOLD PRODUCTION REGION

t–$\bar{t}$ NEAR THRESHOLD PRODUCTION REGION

$E_{cm} \simeq 340 - 360$ **GeV. Nonrelativistic system**

$$v_{top} = \sqrt{1 - \frac{4m_t^2}{s}} \ll 1$$

$$m_t \gg m_t v_{top} \gg m_t v_{top}^2$$

$$\frac{\alpha_s}{v_{top}} \sim 1 \rightarrow \text{Coulomb resummation} \rightarrow \text{Schroedinger equation}$$

$$\text{LO} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v_{top}^n}$$

$$\text{NLO} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v_{top}^n} \times (\alpha_s, v_{top})$$

$$\cdots\cdots$$

t–$\bar{t}$ NEAR THRESHOLD PRODUCTION REGION

$E_{cm} \simeq 340 - 360$ GeV. Nonrelativistic system

$$v_{top} = \sqrt{1 - \frac{4m_t^2}{s}} \ll 1$$

$$m_t \gg m_t v_{top} \gg m_t v_{top}^2$$

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$$\text{LO} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v_{top}^n}$$

$$\text{NLO} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v_{top}^n} \times (\alpha_s, v_{top})$$

$$\ldots\ldots$$

$$\text{LL} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v_{top}^n} \sum_{m=0}^{\infty} d_m \alpha_s^m \ln^m v$$

$$\text{NLL} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v_{top}^n} \sum_{m=0}^{\infty} d_m \alpha_s^m \ln^m v \times (\alpha_s, v_{top})$$

$$\ldots\ldots$$

The top mass

Next Linear Collider. $\delta m_t(\text{exp.}) \lesssim 30$ **MeV; decay width 2%: Martinez-Miquel**

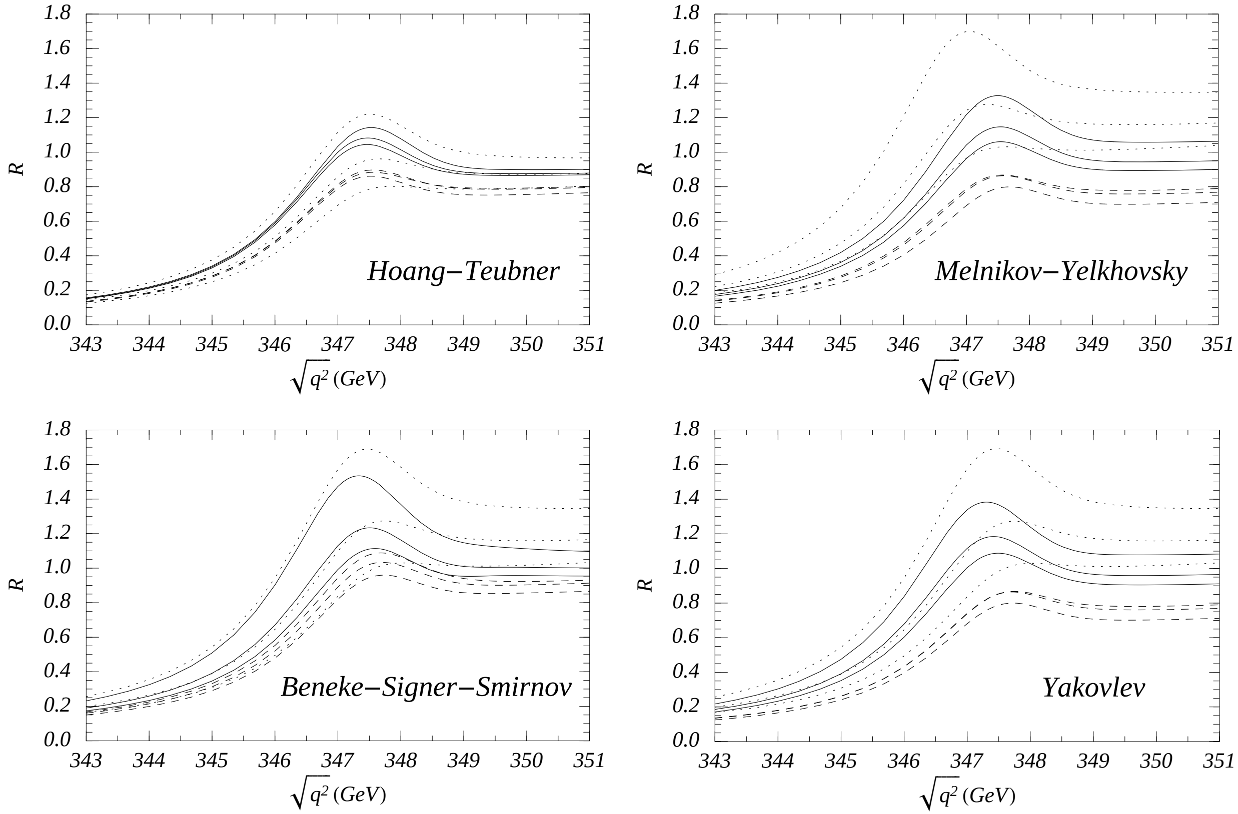


Figure 23:

Total normalized photon induced $t\bar{t}$ cross section at the International Linear Collider versus the center of mass energy at LO (dotted line), NLO (dashed line) and NNLO (solid line). Plot from hep-ph/0001286.

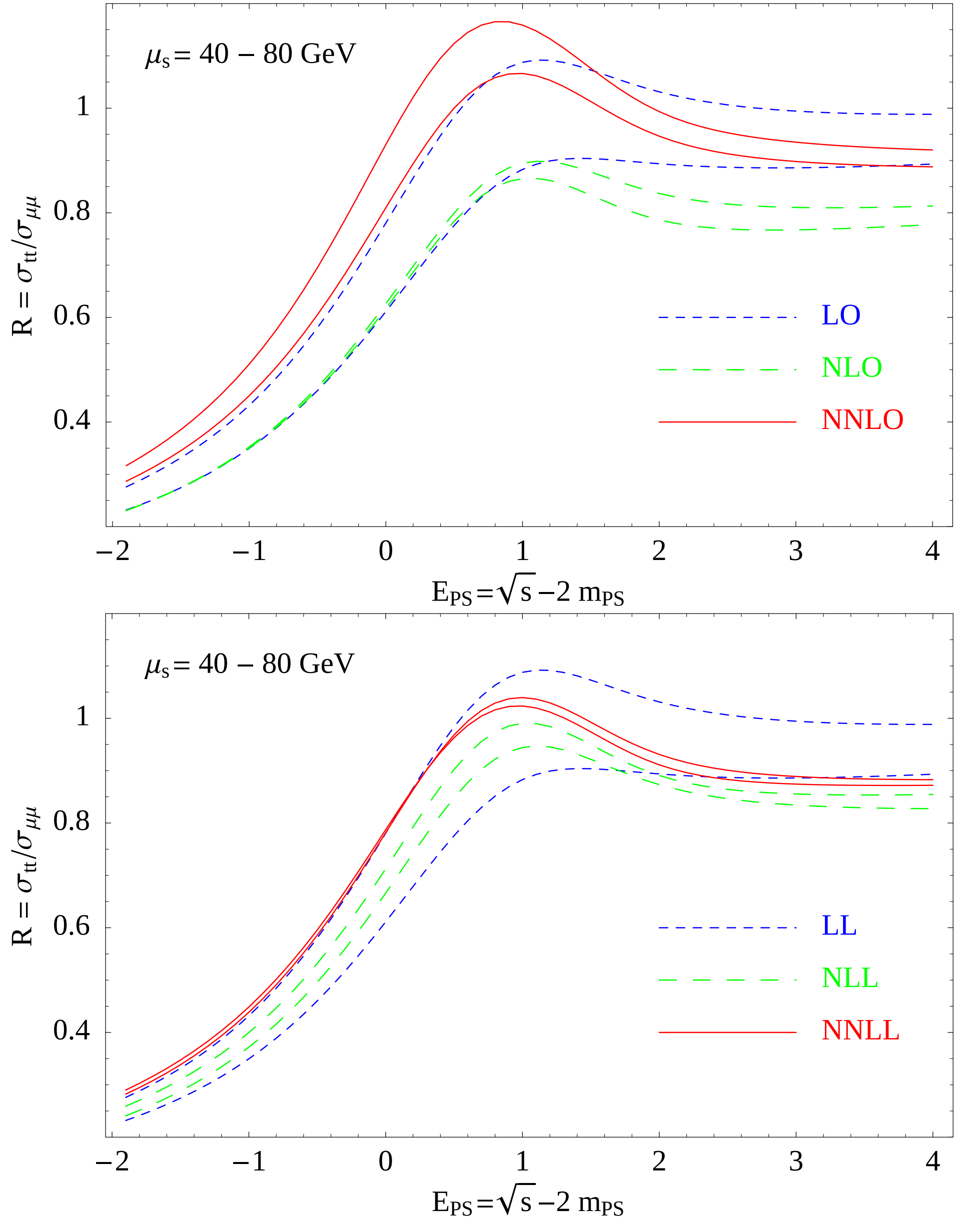
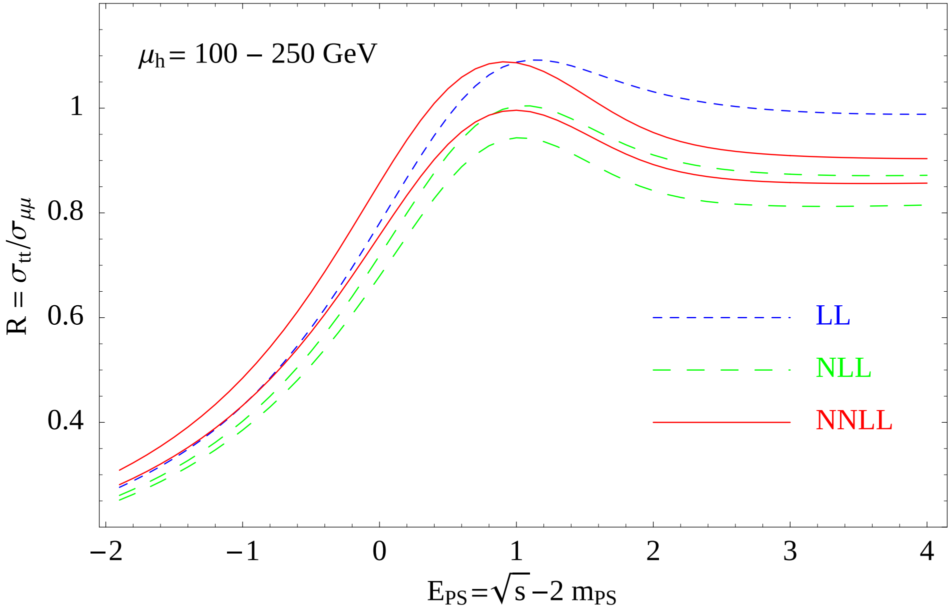


Figure 24: Threshold scan for $t\bar{t}$ using $m_{t,PS} = 175$ GeV. The upper figure shows the fixed order results, LO, NLO and NNLO, whereas the figure belows the RGI results LL, NLL and NNLL are displayed. The soft scale is varied from μ_s=40 GeV to μ_s=80 GeV. Pineda-Signer.

RGI reduces the scale dependence and improves the convergence.



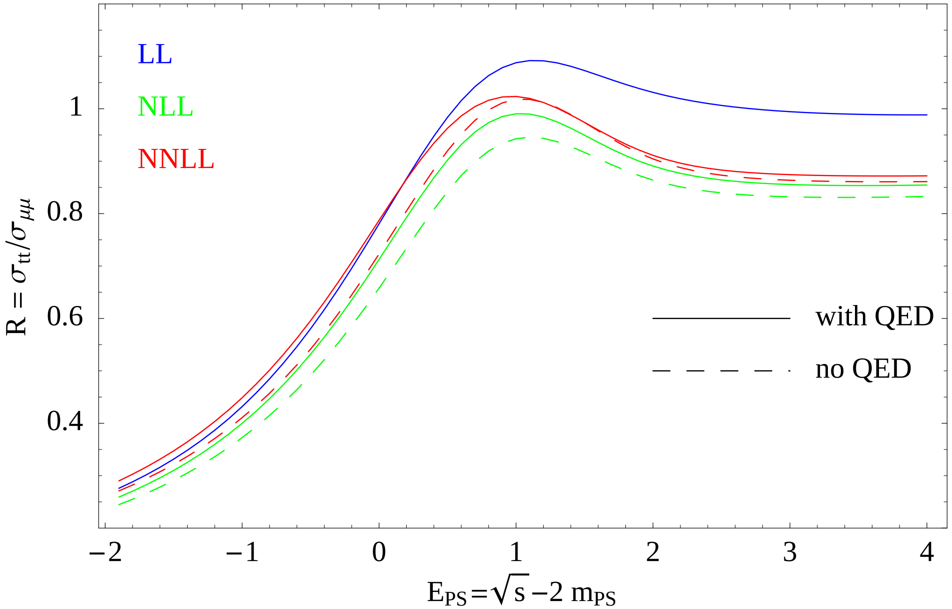


Figure 25: Upper panel: Variation of the hard scale μ_h in the threshold scan for $m_{t,PS} = 175$ GeV. The hard scale is varied from 100 GeV $\leq \mu_h \leq$ 250 GeV. Lower panel: the effect of including QED corrections at NLL and NNLL. There are no QED corrections at LL.

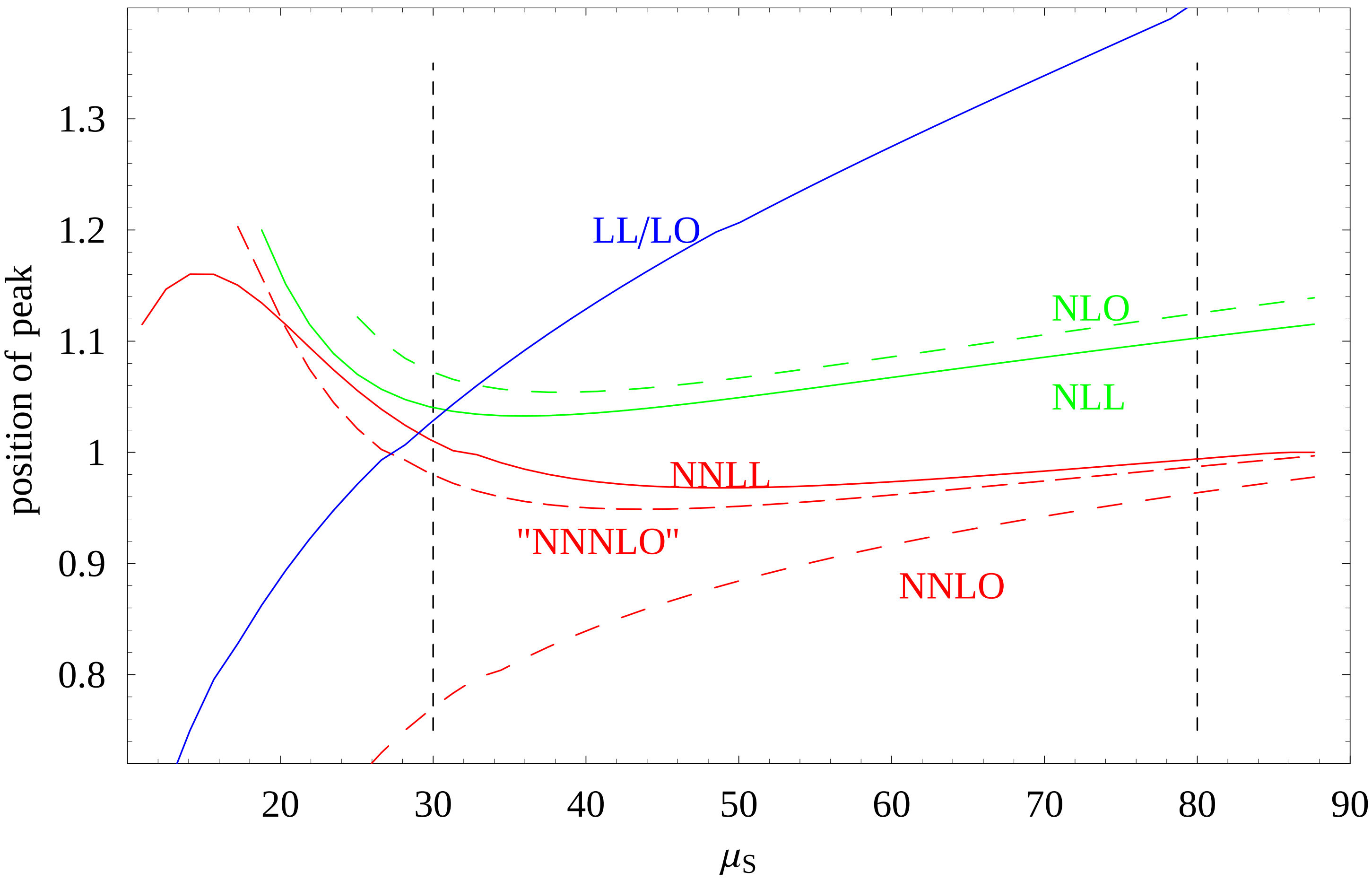


Figure 26: The position of the peak of the RGI threshold cross section as a function of the soft scale μ_s. The vertical dashed lines show the limits of variation used in Figure 24.

Contrary to previous claims, to get an improved determination of the top mass RGI has to be used (or "NNNLO").
Leading logs seem to give the dominant contribution
Strong scale dependence for scales below 30 GeV

BACK UP SLIDES

$\nu \sim m$

Large β_0 analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1 + (2u-1)\ln\frac{\nu}{m} + \cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u - 1 \ll 1$)

$$(2u-1)\ln\frac{\nu}{m} \ll 1\,.$$

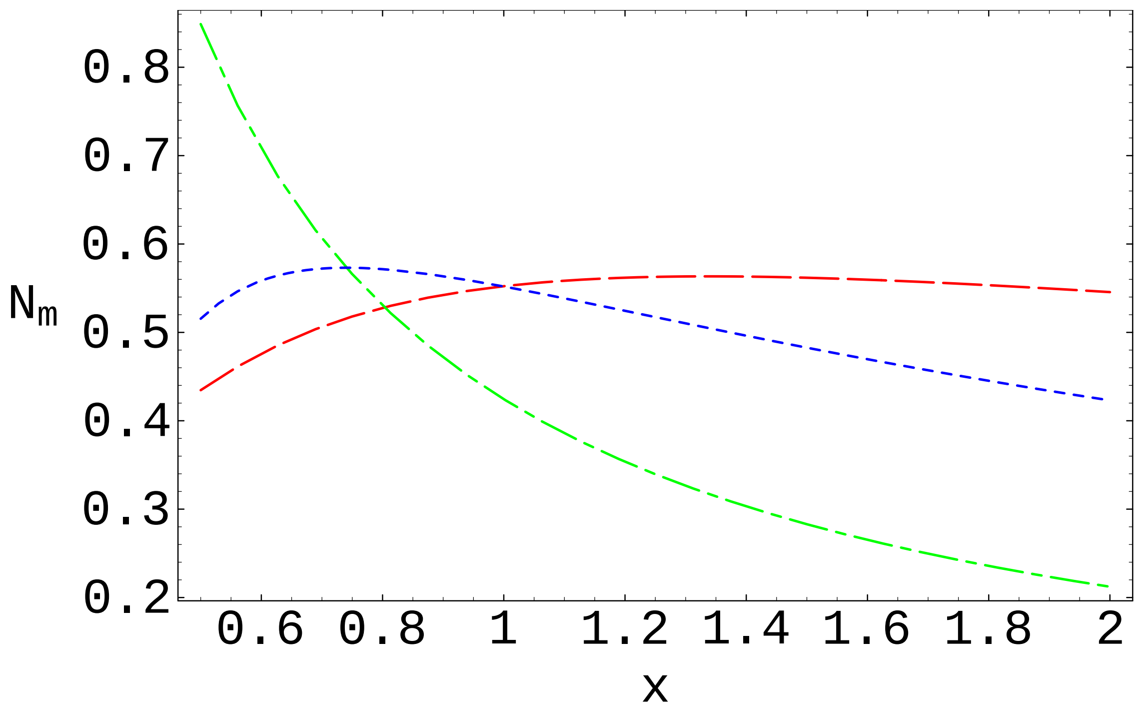


Figure 27: $x \equiv \frac{\nu}{m_{\overline{\rm MS}}}$ *dependence of* N_m *for* $n_f = 4$.

Estimates of r_n

$\tilde{r}_n = r_n/m_{\overline{\text{MS}}}$	$\tilde{r}_0$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$
exact $(n_f = 3)$	0.424413	1.04556	3.75086	$---$	$---$
our estimate $(n_f = 3)$	0.617148	0.977493	3.76832	18.6697	118.441
large β_0 $(n_f = 3)$	0.424413	1.42442	3.83641	17.1286	97.5872
exact $(n_f = 4)$	0.424413	0.940051	3.03854	$---$	$---$
our estimate $(n_f = 4)$	0.645181	0.848362	3.03913	13.8151	80.5776
large β_0 $(n_f = 4)$	0.424413	1.31891	3.28911	13.5972	71.7295
exact $(n_f = 5)$	0.424413	0.834538	2.36832	$---$	$---$
our estimate $(n_f = 5)$	0.706913	0.713994	2.36440	9.73117	51.5952
large β_0 $(n_f = 5)$	0.424413	1.21339	2.78390	10.5880	51.3865

Table 4: *Values of r_n for $\nu = m_{\overline{\text{MS}}}$. Either the exact result (when available), our estimate, or the estimate using the large β_0 approximation.*

$\tilde{r}_n = r_n/m_{\overline{\text{MS}}}$	$\tilde{r}_0$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$
$O(1/n)$ $(n_f = 3)$	-0.164	-0.046	-0.027	-0.019	-0.015
$O(1/n^2)$ $(n_f = 3)$	0.237	-0.103	-0.017	-0.007	-0.004
$O(1/n)$ $(n_f = 4)$	-0.105	-0.028	-0.016	-0.012	-0.009
$O(1/n^2)$ $(n_f = 4)$	0.274	-0.126	-0.020	-0.008	-0.004
$O(1/n)$ $(n_f = 5)$	0.024	0.006	0.003	0.002	0.002
$O(1/n^2)$ $(n_f = 5)$	0.326	-0.165	-0.023	-0.009	-0.005

Table 5: $O(1/n)$ *corrections (normalized with respect the leading solution) of our r_n estimates for different number of light fermions.*

$$c_1(n_f = 0) \simeq -0.215, \quad c_2(n_f = 0) \simeq 0.185$$

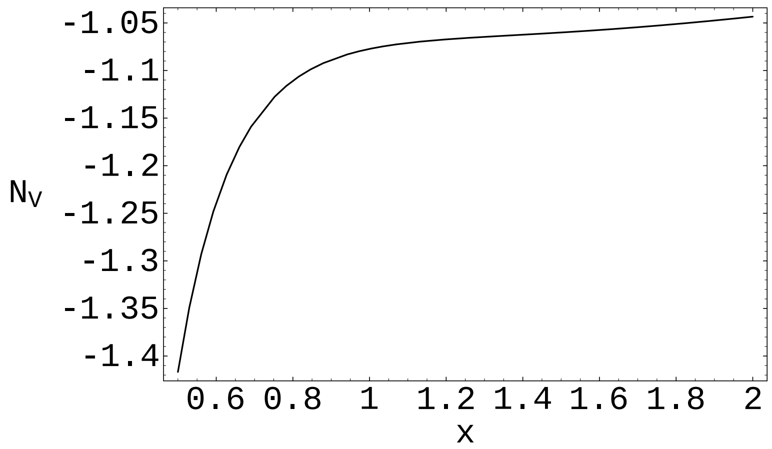


Figure 28: *$x \equiv \nu r$ dependence of N_V for $n_f = 4$.*

$\tilde{V}^{(0)}_{s,n} = rV^{(0)}_{s,n}$	$\tilde{V}^{(0)}_{s,0}$	$\tilde{V}^{(0)}_{s,1}$	$\tilde{V}^{(0)}_{s,2}$	$\tilde{V}^{(0)}_{s,3}$	$\tilde{V}^{(0)}_{s,4}$
exact ($n_f = 3$)	-1.33333	-1.84512	-7.28304	$---$	$---$
our estimate ($n_f = 3$)	-1.23430	-1.95499	-7.53665	-37.3395	-236.882
large β_0 ($n_f = 3$)	-1.33333	-2.69395	-7.69303	-34.0562	$---$
exact ($n_f = 4$)	-1.33333	-1.64557	-5.94978	$---$	$---$
our estimate ($n_f = 4$)	-1.29036	-1.69672	-6.07826	-27.6301	-161.155
large β_0 ($n_f = 4$)	-1.33333	-2.49440	-6.59553	-27.0349	$---$
exact ($n_f = 5$)	-1.33333	-1.44602	-4.70095	$---$	$---$
our estimate ($n_f = 5$)	-1.41383	-1.42799	-4.72881	-19.4623	-103.190
large β_0 ($n_f = 5$)	-1.33333	-2.29485	-5.58246	-21.0518	$---$

Table 6: *Values of $V^{(0)}_{s,n}$ with $\nu = 1/r$. Either the exact result (when available), our estimate, or the estimate using the large β_0 approximation.*