

Lecture 3. Status of renormalons & Resurgence/Nonperturbative-OPE in QCD/YM in \mathbb{R}^4

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Renormalons

Originally (Lautrup, 't Hooft).

Renormalon: summation of "bubbles". Running of α .

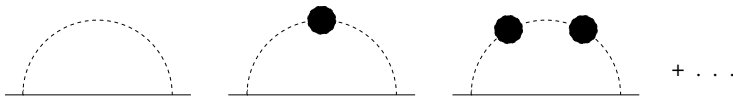


Figure: Sum of the bubbles in the quark propagator.

Pole mass (Bigi, Shifman, Uraltsev, Vainshtein; Beneke, Braun)

$$m_{\text{OS}} = m_{\overline{\text{MS}}} (1 + B_1 \alpha_s + B_2 \alpha_s^2 + \dots) \quad B_n \sim n!$$

$$\delta m \propto \int^\mu dk \alpha(k) \sim \alpha_s(\mu) \sum_{n=0}^{\infty} \left(\frac{\beta_0 \alpha(\mu)}{2\pi} \right)^n \int^\mu dk \ln^n \frac{\mu}{k} = \alpha_s(\mu) \sum_{n=0}^{\infty} \left(\frac{\beta_0 \alpha(\mu)}{2\pi} \right)^n n!$$

$$\beta_0^{\text{QED}} = -\frac{4}{3} T_F n_l \rightarrow \beta_0^{\text{QCD}} = \frac{11}{3} C_A - \frac{4}{3} T_F n_l \text{ naive non-abelianization.}$$

Beyond bubbles: renormalization group methods (Parisi; Beneke, ...)

NP OPE (Novikov, Shifman, Vainshtein, Zakharov)

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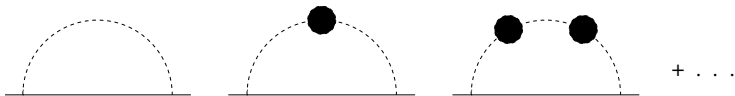


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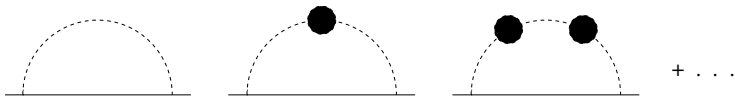


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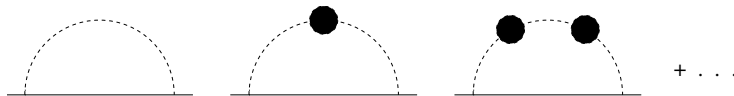


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Modern view: EFT/Factorization

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n \quad c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

The Wilson coefficients are believed to be asymptotic: $c_n \sim n!$

IF SO such behavior should comply with the Operator Product Expansion.

EFT/factorization definition of renormalon: Asymptotic behavior of the perturbative expansion such that the associated ambiguity in the summation of the perturbative series can be absorbed into a higher order operator.

Example:

$$M_B = m_{\text{OS}} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\text{OS}})$$

M_B is renormalon free. Therefore m_{OS} suffers from renormalon ambiguities:

$$m_{\text{OS}} = m_{\overline{\text{MS}}} (1 + B_1 \alpha_s + B_2 \alpha_s^2 + \dots)$$

with $B_n \sim n!$. In other words

$$\delta_{np}^{(\text{pert.})} m_{\text{OS}} = \delta_{np}^{(\text{pert.})} m_{\overline{\text{MS}}} (1 + B_1 \alpha_s + B_2 \alpha_s^2 + \dots) \sim \Lambda_{\text{QCD}}!$$

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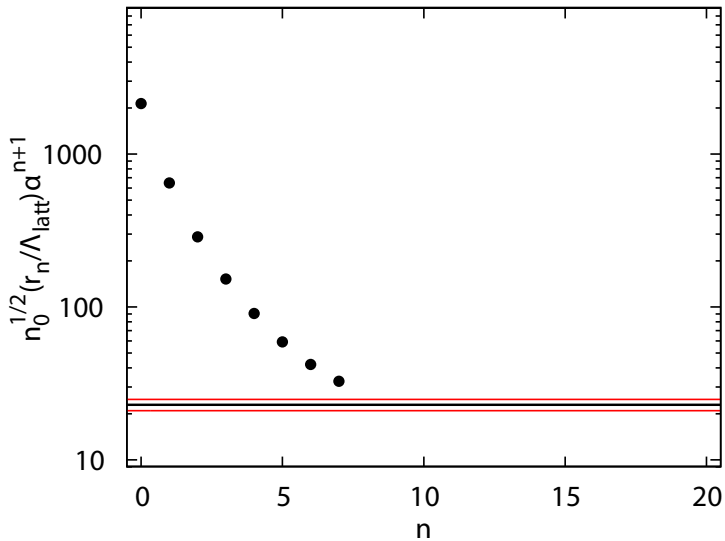
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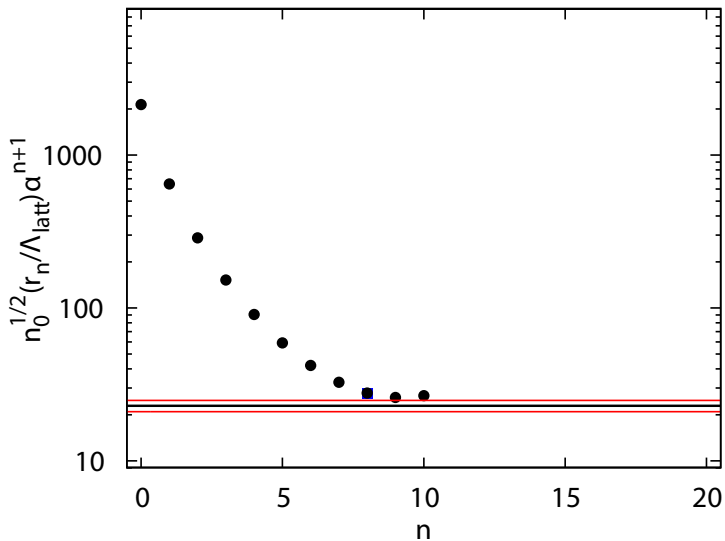
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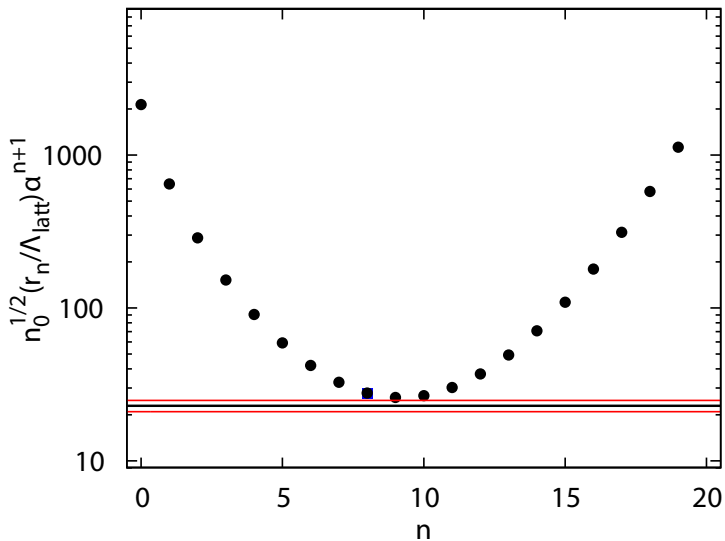
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Maximal accuracy of the Wilson coefficients from a perturbative calculation is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*},$$

where $n^* \sim \frac{a}{\alpha_s}$. If a is positive c suffers from a non-perturbative ambiguity of order

$$\delta c \sim (\Lambda_{\text{QCD}})^{\frac{|a|\beta_0}{2\pi}}.$$

The Borel transform of $c(\nu)$ reads

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^{\infty} dt e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the Wilson coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

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Where do we expect renormalon effects to be more important?

Current-current correlator ($c_n \alpha_s^{n+1} \sim \Lambda_{\text{QCD}}^4 / Q^4 \rightarrow c_n \sim n!$):

$$\int d^4x e^{iqx} \langle \text{vac} | J(x) J(0) | \text{vac} \rangle = (\text{Pert. th.}) + \frac{\Lambda_{\text{QCD}}^4}{Q^4} + \dots$$

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$$\langle P \rangle = (\text{Pert. th.}) + a^4 \Lambda_{\text{QCD}}^4 + \dots$$

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The natural place to look for these effects.

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Bali, AP: hep-ph/0310130

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The large n dependence of r_n is dictated by the closest singularity to the origin of $B[m_{\text{OS}}]$ ($u = \frac{\beta_0 t}{4\pi}$).

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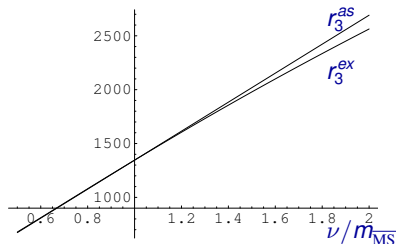
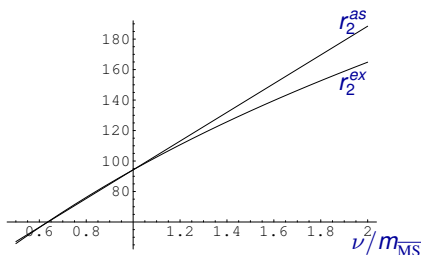
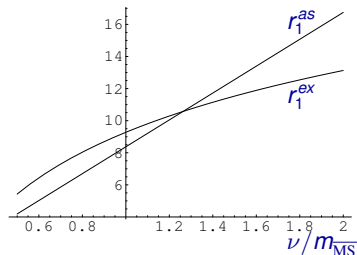
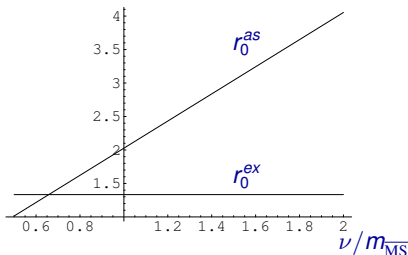
$$\text{Im} \int_0^{\infty} dt e^{-t/\alpha_s} B[m_{\text{OS}}](t) \sim \Lambda_{\text{QCD}} \sim \nu e^{-\frac{2\pi}{\beta_0 \alpha(\nu)}}$$

$$B[m_{\text{OS}}](t) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \dots \right) + (\text{analytic term}),$$

$$r_n^{\text{asym}} = N_m \nu \left(\frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right).$$

$$b = \frac{\beta_1}{2\beta_0^2}, \quad c_1 = \frac{1}{4b\beta_0^3} \left(\frac{\beta_1^2}{\beta_0} - \beta_2 \right), \quad \dots$$

Check : $r_n \stackrel{n \rightarrow \infty}{\sim} m_{\overline{\text{MS}}} \left(\frac{\beta_0}{2\pi} \right)^n n! N_m \sum_{s=0}^n \frac{\ln^s[\nu/m_{\overline{\text{MS}}}]}{s!} \sim \nu$



Determination of N_m

$$\nu \sim m$$

Large β_0 analysis

$$m \left(\frac{\nu}{m} \right)^{2u} \simeq \nu \{ 1 + (2u - 1) \ln \frac{\nu}{m} + \dots \}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u - 1 \ll 1$)

$$(2u - 1) \ln \frac{\nu}{m} \ll 1.$$

$$N_m = \frac{r_n}{(r_n^{\text{asym}} / N_m)}$$

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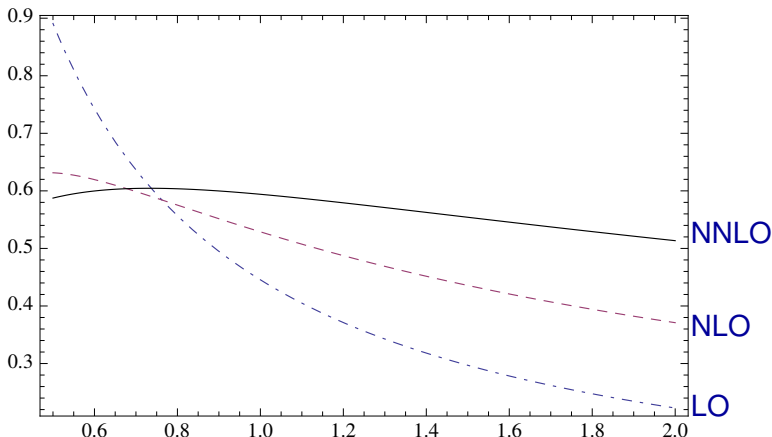


Figure: N_m for $n_f = 3$, as a function of $x \equiv \mu/m_b$, obtained from r_n/r_n^{asym} with r_n^{asym} truncated at $\mathcal{O}(1/n^3)$. We name the different lines as NLO (dashed-dotted), NLO (dashed) and NNLO (solid) for $n = 0, 1, 2$, respectively.

The static potential

$$V(r; \nu_{us}) = \sum_{n=0}^{\infty} V_n \alpha_s^{n+1},$$

$2m_{0S} + V$ can be understood as an observable up to $O(r^2 \Lambda_{\text{QCD}}^3, \Lambda_{\text{QCD}}^2/m)$ contributions $\rightarrow 2N_m + N_V = 0$ and

$$V_n^{asym} = N_V \nu \left(\frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right)$$

$$N_V = \frac{V_n}{(V_n^{asym}/N_V)}$$

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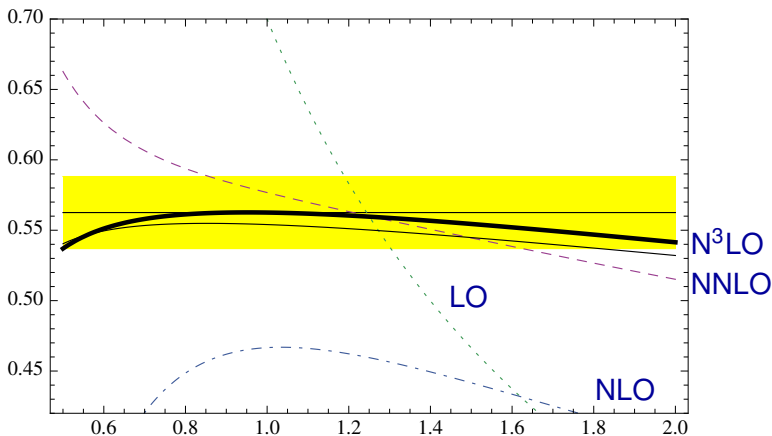


Figure: $-N_V/2 = N_m$ for $n_l = 3$, as a function of $x \equiv \nu r$, obtained from $-(N_V/2)v_n/v_n^{asym}$. v_n^{asym} is truncated at $\mathcal{O}(1/n^3)$.

$$N_m(n_l = 0) = 0.600(29) ,$$

$$N_m(n_l = 3) = 0.563(26) .$$

~ 20 standard deviations from zero!

Conformal window: n_f dependence

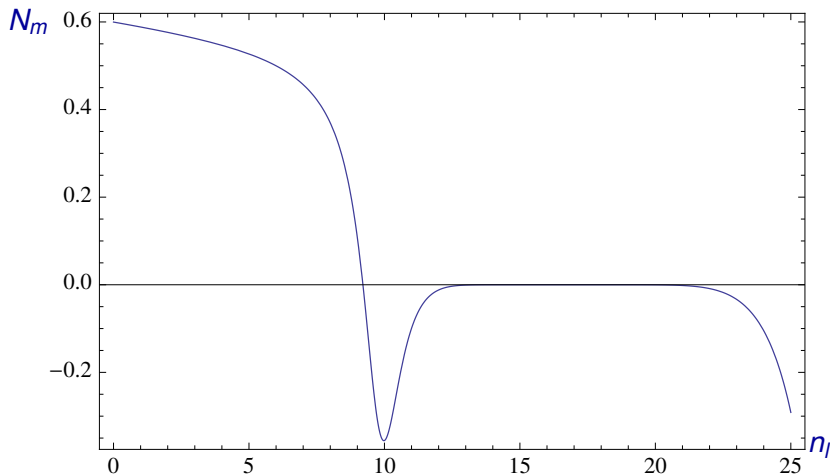


Figure: $N_m(x = 1)$ obtained from $-(N_V/2)v_3/v_3^{asym}$ as a function of n_f .

First numerical evidence of the disappearance of the renormalon in the conformal window.

Yet...

- ▶ Not possible to compute using known semiclassical analysis.
- ▶ Based on few orders in perturbation theory ($\sim 3, 4$)
- ▶ Claims against renormalon existence (Suslov), or against renormalon relevance (implicit in sum rules).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946

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POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the **lattice**: δm of HQET.
 We use **Numerical Stochastic Perturbation Theory** (Di Renzo et al.).

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_n \frac{1}{d_R} \text{tr} \left[\prod_{n_4=0}^{N_T-1} U_4^R(n) \right] \quad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$\langle L^{(R)}(N_S, N_T) \rangle \stackrel{N_T \rightarrow \infty}{\sim} e^{-N_T a \delta m^{(R)}(N_S)}$$

$$\delta m^{(R)}(N_S) = \lim_{N_T \rightarrow \infty} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S, N_T) \alpha^{n+1} = \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S) \alpha^{n+1} (1/a)$$

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"Physical interpretation"

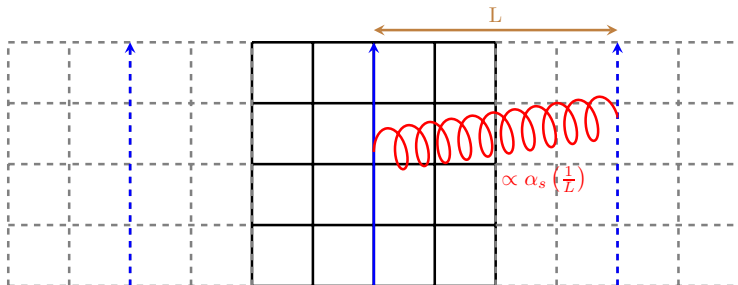


Figure: *Self-interactions with replicas producing $1/L = 1/(aN_S)$ Coulomb terms.*

$$P \propto \int_{1/(aN_S)}^{1/a} dk \alpha(k) \sim \frac{1}{a} \sum_n c_n \alpha^{n+1} (a^{-1}) - \frac{1}{aN_S} \sum_n c_n \alpha^{n+1} ((aN_S)^{-1}),$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi} \right)^n n!, \quad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi} \right)^n \frac{n!}{i!}.$$

Perturbative OPE (Zimmermann) at finite volume ($N_S \rightarrow \infty$)

$$\delta m = \lim_{N_S \rightarrow \infty} \delta m(N_S) \quad c_n = \lim_{N_S \rightarrow \infty} c_n(N_S) \quad \left(\lim_{n \rightarrow \infty} c_n^{(3,\rho)} = r_n(\nu)/\nu \right).$$

For large N_S , we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a}$)

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Taylor expansion of $\alpha((aN_S)^{-1})$ in powers of $\alpha(a^{-1})$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \quad f_n(N_S) = \sum_{i=0}^n f_n^{(i)} \ln^i(N_S),$$

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and so on.

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

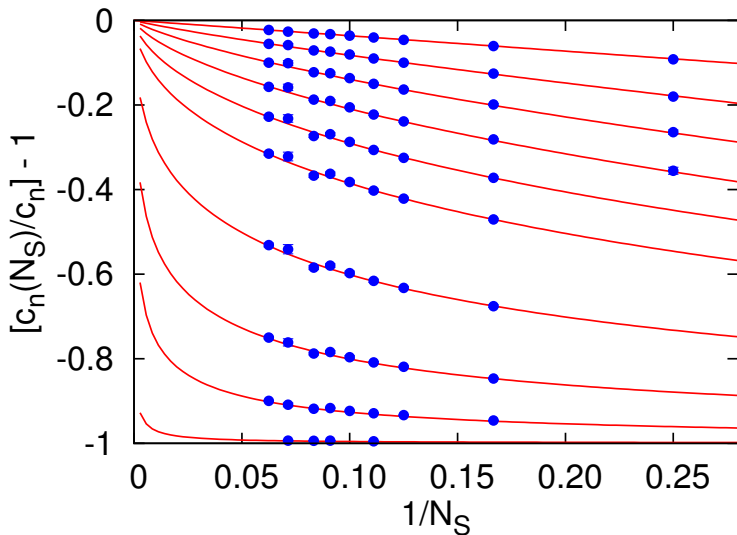


Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ for $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ (top to bottom). For each value of N_S we have plotted the data point with the maximum value of N_T . The curves represent the global fit. $-(1/N_S)f_{0,\text{DLPT}}^{(3,0)}/c_{0,\text{DLPT}}^{(3,0)}$ is shown for $n = 0$.

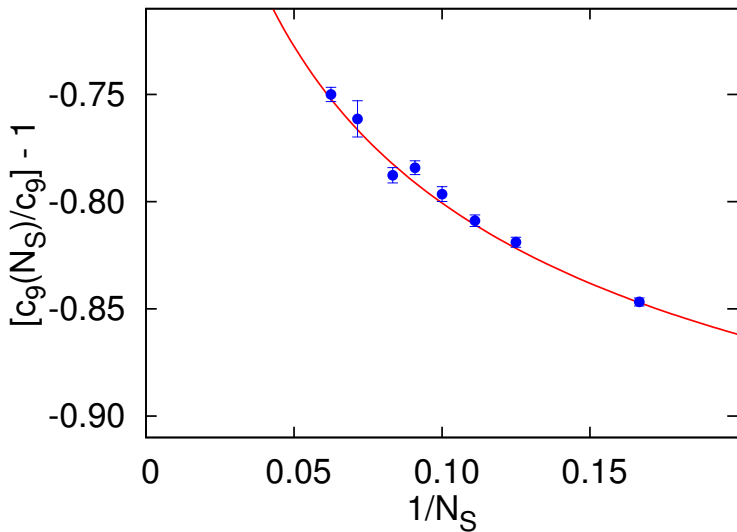
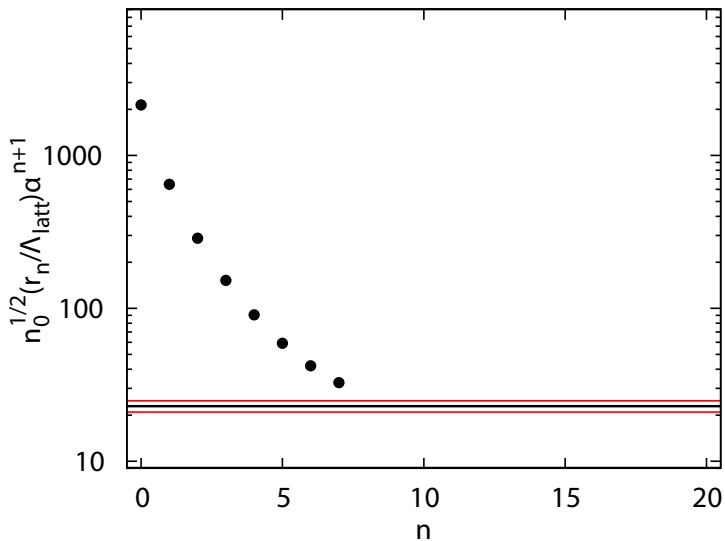
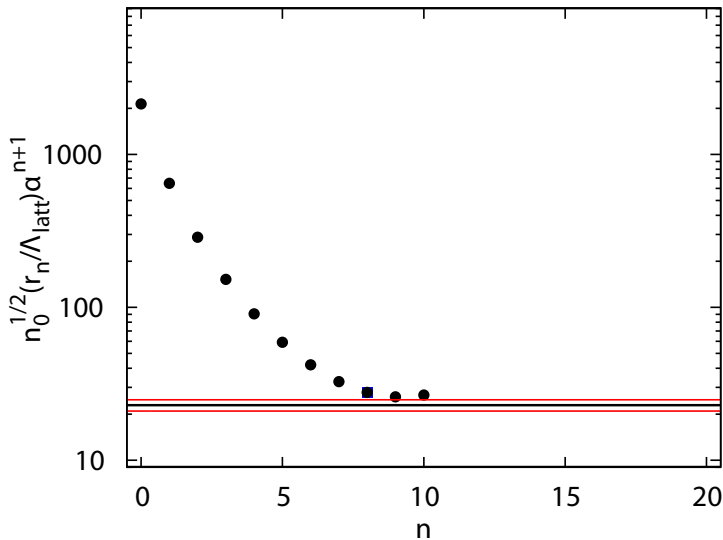
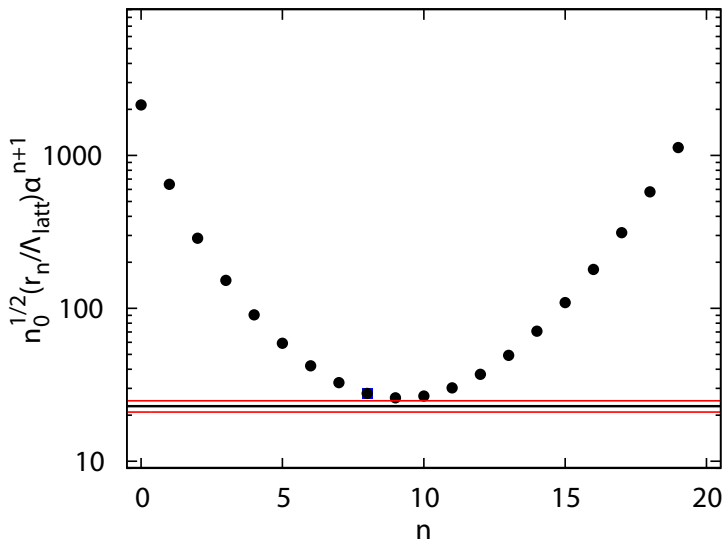


Figure: Zoom of previous Figure for $n = 9$.







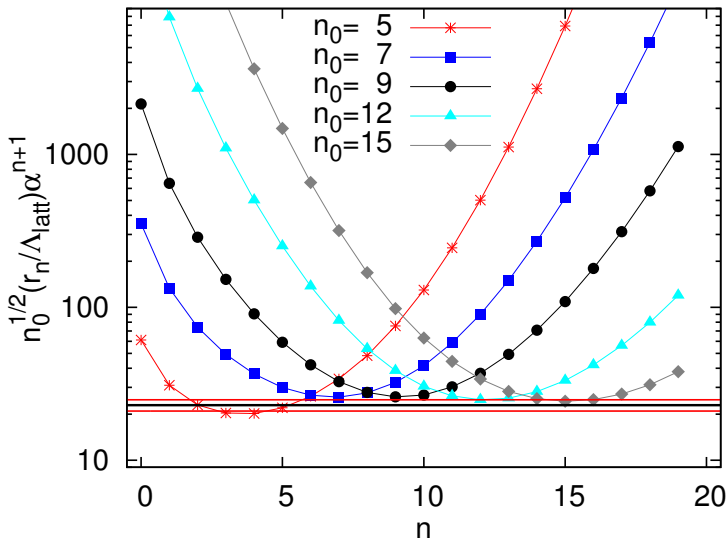


Figure: c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α , ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$). Bali, Bauer, AP, Torrero, 1303.3279.

Ratios

$$\begin{aligned} \frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}} \frac{1}{n} &= \frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}} \frac{1}{n} \\ &= \frac{\beta_0}{2\pi} \left\{ 1 + \frac{b}{n} - \frac{bs_1}{n^2} + \frac{1}{n^3} \left[b^2 s_1^2 + b(b-1)(s_1 - 2s_2) \right] + \mathcal{O}\left(\frac{1}{n^4}\right) \right\} . \end{aligned}$$

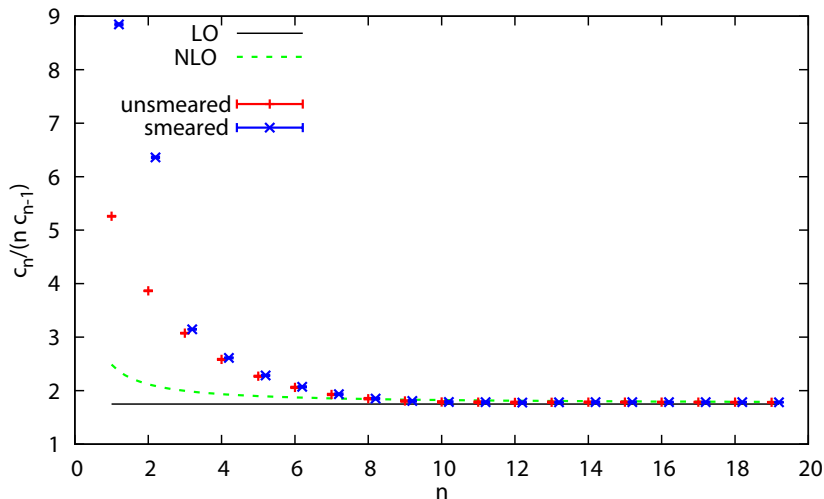


Figure: Ratios $c_n / (n c_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

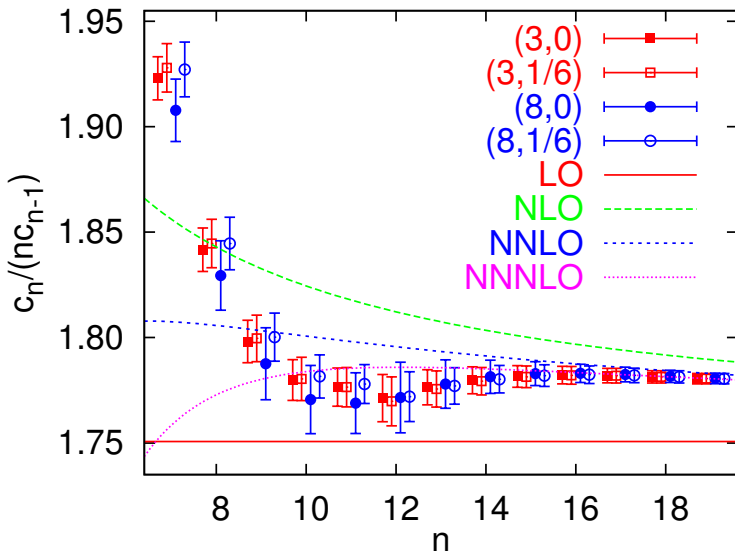


Figure: The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.

N_m

$$c_n^{\text{fitted}} = N_m \left(\frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right).$$

$$f_n^{\text{fitted}} = N_m \left(\frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right).$$

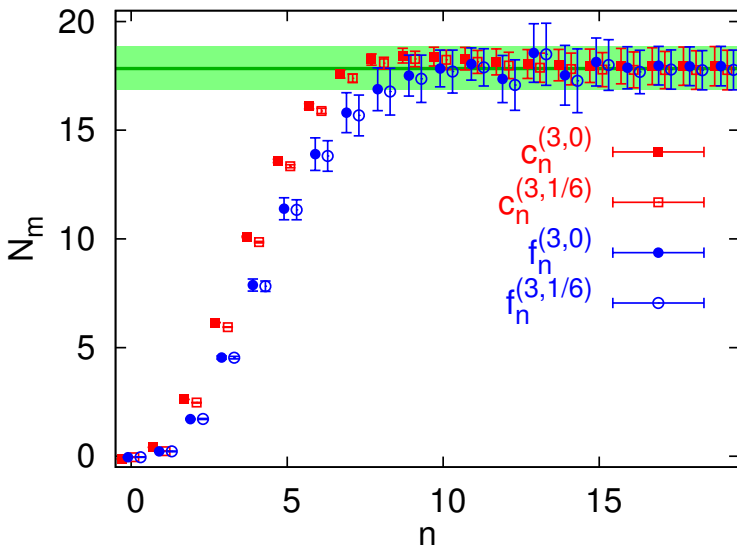


Figure: N_m^{latt} determined via r_n truncated at NNNLO, from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result: $N_m = 17.9(1.0)$

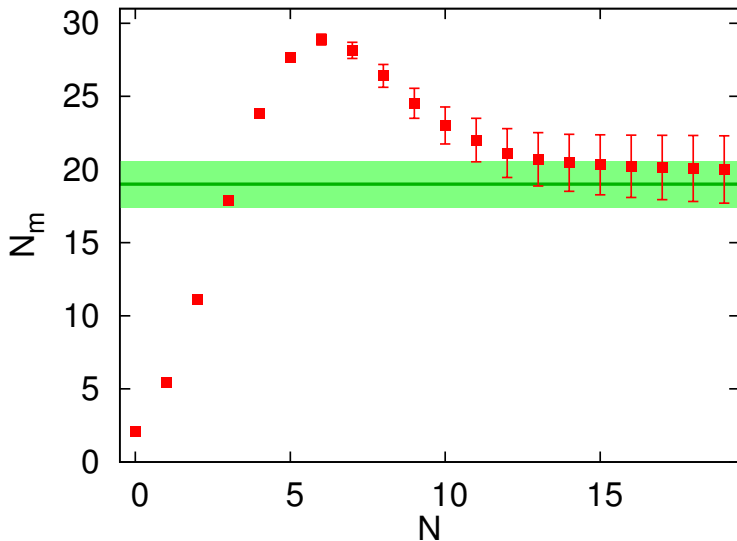


Figure: N_m^{latt} , determined from the coefficients $c_n^{(3,0)}$, $n \leq N$, using $(1 - 2u)^{1+b} B^{(N)}[\delta m](t(u))$.

From lattice to $\overline{\text{MS}}$ scheme

$$\alpha_{\overline{\text{MS}}}(\mu) = \alpha_{\text{latt}}(\mu) \left(1 + d_1 \alpha_{\text{latt}}(\mu) + d_2 \alpha_{\text{latt}}^2(\mu) + d_3 \alpha_{\text{latt}}^3(\mu) + \mathcal{O}(\alpha_{\text{latt}}^4) \right),$$

$$N_{m, m_{\bar{g}}}^{\overline{\text{MS}}} = N_{m, m_{\bar{g}}}^{\text{latt}} \Lambda_{\text{latt}} / \Lambda_{\overline{\text{MS}}}, \quad \text{where} \quad \Lambda_{\overline{\text{MS}}} = e^{\frac{2\pi d_1}{\beta_0}} \Lambda_{\text{latt}} \approx 28.809338139488 \Lambda_{\text{latt}}.$$

This yields the numerical values

$$N_m^{\overline{\text{MS}}} = 0.620(35), \quad C_F/C_A N_{m_{\bar{g}}}^{\overline{\text{MS}}} = -C_F/C_A N_{\Lambda}^{\overline{\text{MS}}} = 0.610(41).$$

Before $N_m^{\overline{\text{MS}}} = 0.600(29)$. Combined $N_m^{\overline{\text{MS}}} = 0.608(22)$.

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Plaquette (Bali, Bauer, AP: 1401.7999, 1403.6477)

$$\langle P \rangle_{\text{pert}}(N) \equiv \frac{1}{Z} \int [dU_{x,\mu}] e^{-S[U]} P[U] \Big|_{\text{NSPT}} = \sum_{n \geq 0} p_n(N) \alpha^{n+1}$$

Perturbative OPE

$$\frac{1}{a} \gg \frac{1}{Na} \rightarrow \langle P \rangle_{\text{pert}}(N) = P_{\text{pert}}(\alpha) \langle 1 \rangle + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle G^2 \rangle_{\text{soft}} + \mathcal{O}\left(\frac{1}{N^6}\right),$$

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$$\begin{aligned} \langle P \rangle_{\text{pert}}(N) &= \sum_{n \geq 0} \left[p_n - \frac{f_n(N)}{N^4} \right] \alpha^{n+1} \\ &= \sum_{n \geq 0} p_n \alpha^{n+1} - \frac{1}{N^4} \left(1 + \sum_{k \geq 0} c_k \alpha^{k+1} (a^{-1}) \right) \times \sum_{n \geq 0} f_n \alpha^{n+1} ((Na)^{-1}) + \mathcal{O}\left(\frac{1}{N^6}\right), \\ \left(\delta m(N_S) \right) &= \frac{1}{a} \sum_{n=0}^{\infty} c_n \alpha^{n+1} (a^{-1}) - \frac{1}{aN_S} \sum_{n=0}^{\infty} f_n \alpha^{n+1} ((aN_S)^{-1}) + \mathcal{O}\left(\frac{1}{N_S^2}\right) \end{aligned}$$

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$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

(before Di Renzo et al. $N+1=8$; Horsley et al. $N+1=20$)

Renormalon expectations:

$$p_n^{\text{latt}} \stackrel{n \rightarrow \infty}{\simeq} N_P^{\text{latt}} \left(\frac{\beta_0}{2\pi d} \right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)} \left\{ 1 + \frac{20.09}{n+db} + \frac{505 \pm 33}{(n+db)^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \right\}.$$

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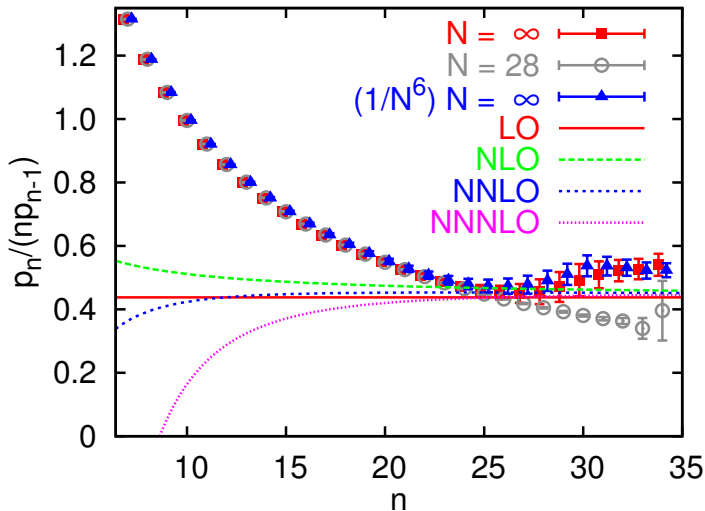


Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients p_n ($N = \infty$, $N = 28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

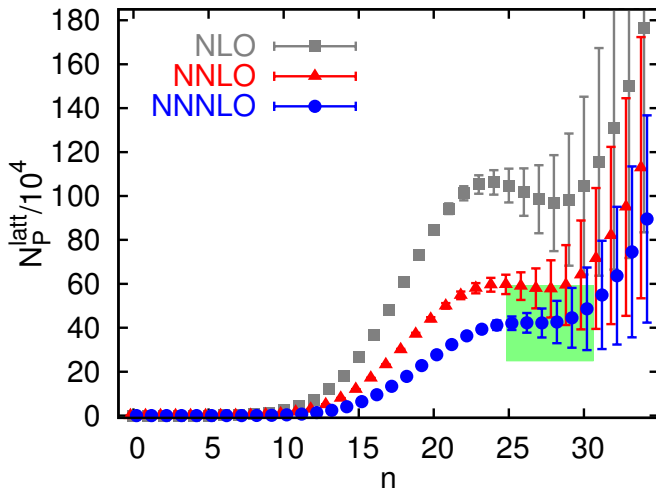


Figure: N_P , determined from the coefficients p_n truncated at NLO, NNLO and NNNLO. The green box marks our final result.

$$N_P^{\overline{\text{MS}}} = 0.61(25) \quad N_G^{\overline{\text{MS}}} = \frac{36}{\pi^2} N_P^{\overline{\text{MS}}} = 2.24(92).$$

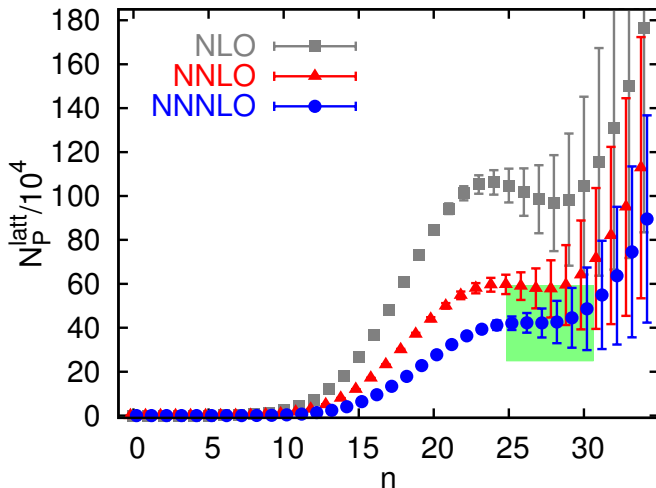


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Beyond perturbation theory (at last...)

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$$\frac{1}{a} \gg \frac{1}{Na}$$

$$\langle P \rangle_{\text{MC}} = \frac{1}{Z} \int [dU_{x,\mu}] e^{-S[U]} P[U] \Big|_{\text{MC}} = P_{\text{pert}}(\alpha) \langle 1 \rangle + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle G^2 \rangle_{\text{MC}} + \mathcal{O}(a^6).$$

$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\text{QCD}} \rightarrow \langle G^2 \rangle_{\text{MC}} = \langle G^2 \rangle_{\text{soft}} \left[1 + \mathcal{O}(\Lambda_{\text{QCD}}^2 (Na)^2) \right]$$

$$\frac{1}{a} \gg \Lambda_{\text{QCD}} \gg \frac{1}{Na} \rightarrow \langle G^2 \rangle_{\text{MC}} = \langle G^2 \rangle_{\text{NP}} \left[1 + \mathcal{O} \left(\frac{1}{\Lambda_{\text{QCD}}^2 (Na)^2} \right) \right],$$

where $\langle G^2 \rangle_{\text{NP}} \sim \Lambda_{\text{QCD}}^4$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

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$$\frac{1}{a} \gg \Lambda_{\text{QCD}} \gg \frac{1}{Na} \rightarrow \langle G^2 \rangle_{\text{MC}} = \langle G^2 \rangle_{\text{NP}} \left[1 + \mathcal{O} \left(\frac{1}{\Lambda_{\text{QCD}}^2 (Na)^2} \right) \right],$$

where $\langle G^2 \rangle_{\text{NP}} \sim \Lambda_{\text{QCD}}^4$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\text{QCD}}$$

$$\langle P \rangle_{\text{MC}} = \frac{1}{Z} \int [dU_{x,\mu}] e^{-S[U]} P[U] \Big|_{\text{MC}} = P_{\text{pert}}(\alpha) \langle 1 \rangle + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle G^2 \rangle_{\text{MC}} + \mathcal{O}(a^6).$$

$$\langle G^2 \rangle_{\text{MC}} = \langle G^2 \rangle_{\text{soft}} \left[1 + \mathcal{O}(\Lambda_{\text{QCD}}^2 (Na)^2) \right], \quad \frac{\pi^2}{36} a^4 \langle G^2 \rangle_{\text{soft}} = -\frac{1}{N^4} \sum_{n \geq 0} f_n \alpha^{n+1} ((Na)^{-1})$$

In this limit non-perturbative effects can be computed at weak coupling (still far from trivial: see resurgence analysis in 1+1 dimensions).

Observations:

- ▶ In this limit the gluon condensate renormalon is not produced by non-perturbative effects.
- ▶ The resummation of all $(\Lambda_{\text{QCD}}^2 (Na)^2)^n$ effects remains to be done to reach the scaling region at infinite volume: $\frac{1}{a} \gg \Lambda_{\text{QCD}} \gg \frac{1}{Na}$, i.e. $\langle G^2 \rangle_{\text{NP}}$.

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Determination of the gluon condensate: $\frac{1}{a} \gg \Lambda_{\text{QCD}} \gg \frac{1}{Na}$

$$\langle G^2 \rangle_{\text{NP}} = \frac{36C_G^{-1}(\alpha)}{\pi^2 a^4(\alpha)} [\langle P \rangle_{\text{MC}}(\alpha) - S_P(\alpha)] + \mathcal{O}(a^2 \Lambda_{\text{QCD}}^2).$$

$$S_P(\alpha) \equiv S_{n_0}(\alpha), \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^n p_j \alpha^{j+1}.$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0} \alpha^{n_0+1}$ is minimal.

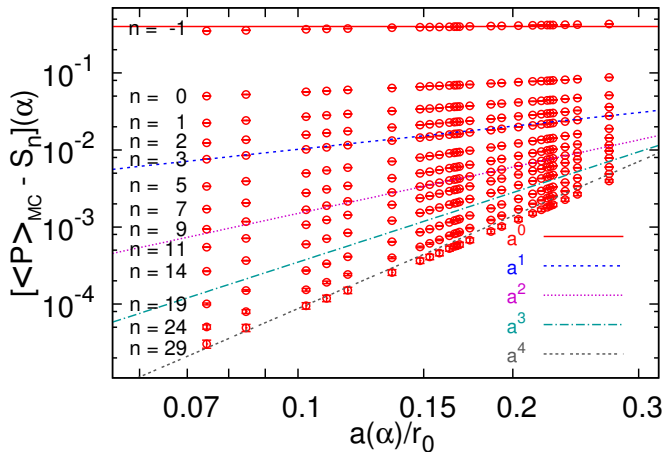


Figure: $\langle P \rangle_{\text{MC}}(\alpha) - S_n(\alpha)$ between MC data and sums truncated at orders α^{n+1} ($S_{-1} = 0$) vs. $a(\alpha)/r_0$. The lines $\propto a^j$ are drawn to guide the eye.

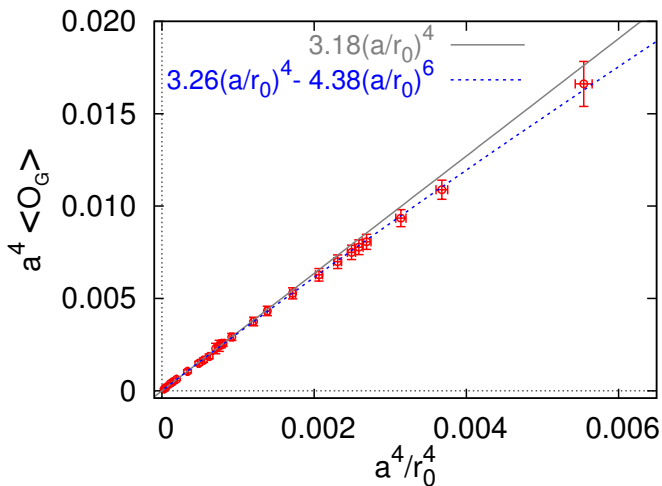


Figure: $\langle P \rangle_{MC}(\alpha) - S_P(\alpha)$. The linear fit is to $a^4 < 0.0013 r_0^4$ points only.

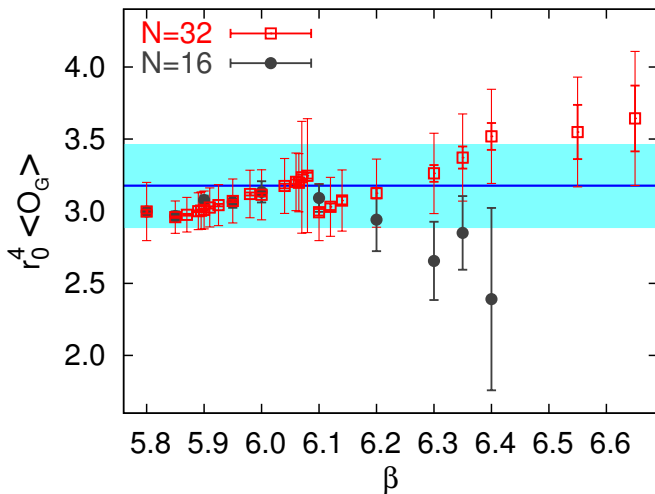


Figure: $\langle G^2 \rangle$ evaluated using the $N = 16$ and $N = 32$ MC data of Boyd et al. The error band is our prediction for $\langle G^2 \rangle$.

$$\langle G^2 \rangle = 3.18(29)r_0^{-4} = 24.2(8.0)\Lambda_{\overline{MS}}^4 \simeq 0.077 \text{ GeV}^4.$$

Uncertainty of the sum due to the truncation

$$\delta S_P = \sqrt{n_0} p_{n_0} \alpha^{n_0+1} \approx \frac{(2\pi)^{3/2} d^{1+db}}{2^{db} \beta_0 \Gamma(1+db)} N_P(\Lambda a)^4 \approx 12.06 N_P(\Lambda a)^4.$$

This object is scheme- and scale-independent (to 1/n-precision)

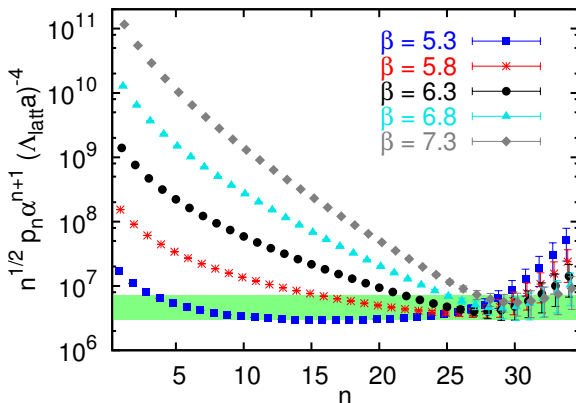


Figure: $\sqrt{n} p_n \alpha^{n+1} / (\Lambda_{\text{latt}} a)^4$, versus n for $\beta = 5.3, 5.8, 6.3, 6.8$ and 7.3 . The green band is the theoretical expectation $12.06 N_P = 5.1(2.1) \times 10^6$.

$$\sqrt{n_0} \frac{|r_{n_0}|}{\Lambda_{\text{latt}}} \alpha^{n_0+1}(\nu) = \frac{2^{3/2-b} \pi^{3/2}}{\beta_0 \Gamma(1+b)} |N_m| \approx 1.206 |N_m|,$$

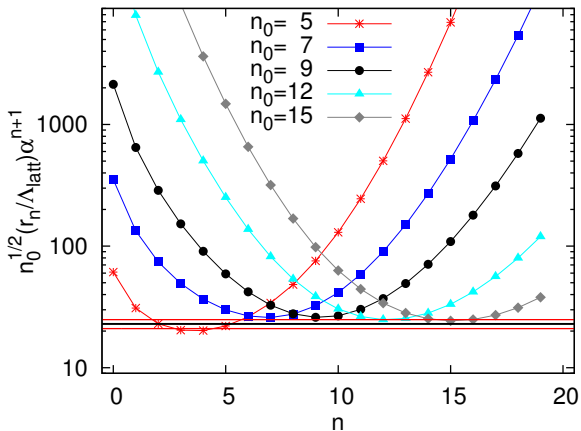


Figure: c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α , ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$).

$$\delta\langle G^2 \rangle_{\text{NP}} \simeq \frac{(2\pi)^{3/2} d^{1+db}}{2^{db} \beta_0 \Gamma(1+db)} N_G^{\overline{\text{MS}}} \Big|_{n_f=0} \Lambda_{\overline{\text{MS}}}^4 = 27(11) \Lambda_{\overline{\text{MS}}}^4 \sim 0.087 \text{ GeV}^4.$$

$$\langle G^2 \rangle = 3.18(29) r_0^{-4} = 24.2(8.0) \Lambda_{\overline{\text{MS}}}^4 \simeq 0.077 \text{ GeV}^4.$$

CONCLUSIONS

Renormalons go beyond large- β_0 analysis: \rightarrow (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\text{MS}}$ -like computations: Pole mass, static potential, \dots

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.600(29), \quad N_m^{\overline{\text{MS}}}(n_l = 3) = 0.563(26).$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20} , vastly increasing the credibility of the prediction.

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.620(35), \quad C_F/C_A N_\Lambda^{\overline{\text{MS}}}(n_l = 0) = -0.610(41).$$

Two independent determinations with very different systematics.

We have (numerically) proven, beyond any reasonable doubt (~ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

First evidence of disappearance of renormalon in the conformal window.

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$$N_P^{\overline{\text{MS}}} = 0.61(25) \quad N_G^{\overline{\text{MS}}} = \frac{36}{\pi^2} N_P^{\overline{\text{MS}}} = 2.24(92).$$

Nonperturbative quantities ($\bar{\Lambda}$, Λ_H , $\langle G^2 \rangle$, ...) can only be defined after subtracting the divergent perturbative series.

$$\delta \langle G^2 \rangle_{\text{NP}} = 27(11) \Lambda_{\overline{\text{MS}}}^4 \simeq 0.087 \text{ GeV}^4. \quad \langle G^2 \rangle = 24.2(8.0) \Lambda_{\overline{\text{MS}}}^4 \simeq 0.077 \text{ GeV}^4.$$

Resurgence/Non-perturbative OPE OK (for the plaquette)

$$\langle P \rangle_{\text{MC}}(\alpha) \simeq \sum_{j=0}^{\eta_0(\alpha)} \rho_n \alpha^{j+1} + \# \left[\alpha^{-\frac{\beta_1}{2\beta_0^2}} e^{-\frac{2\pi}{\beta_0\alpha}} \right]^4 + \dots$$

No Dimension-two condensates: artifacts of incomplete subtractions

- ▶ unquantifiable error due to the simplified parameterization of higher order perturbation theory
- ▶ short distance effect \rightarrow process dependent

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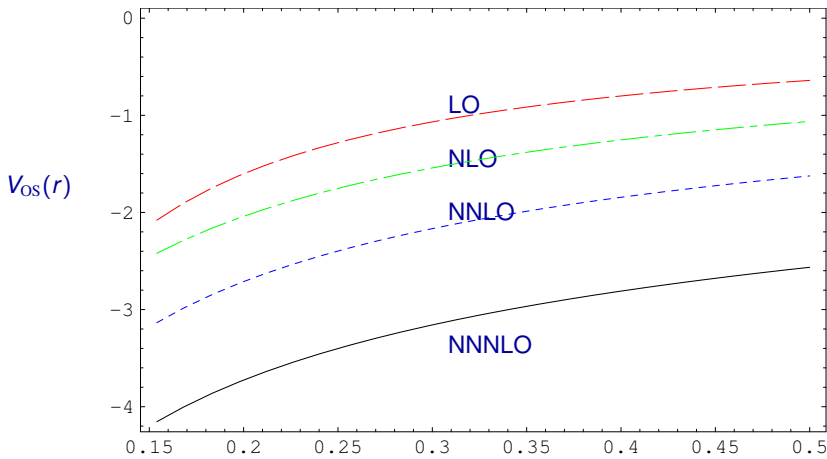
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The static singlet potential

The introduction of renormalons allows to obtain agreement between lattice simulations and perturbation theory.

$$E_s = 2m_{\text{OS}} + V_{\text{s,OS}} + \mathcal{O}(r^2)$$



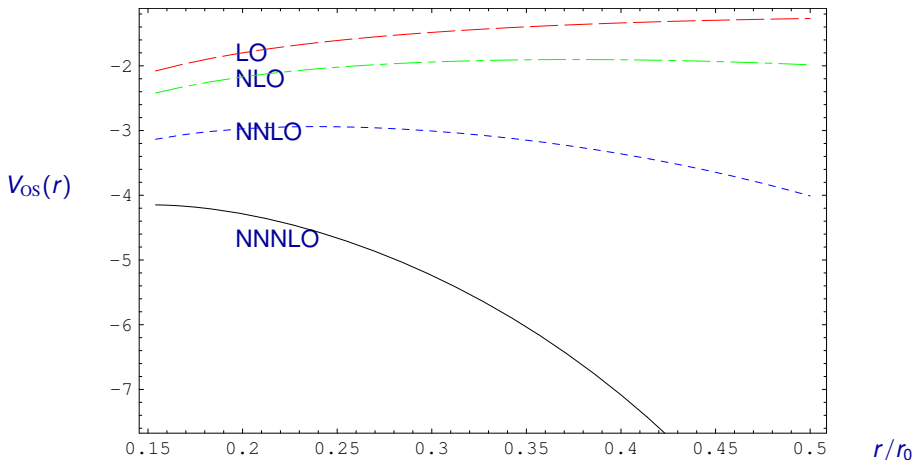


Figure: Plot of $V_{OS}(r)$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line). For the scale of $\alpha_S(\nu)$, we set $\nu = 1/r$. $\nu_{US} = 2.5 r_0^{-1}$.

$$V_{OS}(r) - V_{OS}(r') + E_{latt.}(r')$$

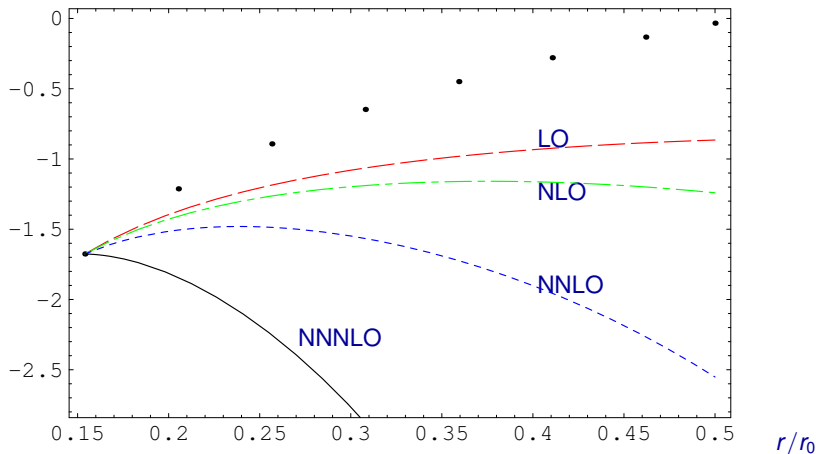


Figure: Plot of $V_{OS}(r) - V_{OS}(r') + E_{latt.}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_S(\nu)$, we set $\nu = 1/r$. $\nu_{US} = 2.5 r_0^{-1}$.

$$E_s = 2m_{\text{RS}}(\nu_f) + V_{s,\text{RS}}(\nu_f) + \mathcal{O}(r^2)$$

$$m_{\text{RS}}(\nu_f) = m_{\text{OS}} - \sum_{n=0}^{\infty} N_n \nu_f \left(\frac{\beta_0}{2\pi} \right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}.$$

Expansion in $\alpha_s(\nu)$

$$m_{\text{RS}}(\nu_f) = m_{\overline{\text{MS}}} + \sum_{n=0}^{\infty} r_n^{\text{RS}} \alpha_s^{n+1},$$

where $r_n^{\text{RS}} = r_n^{\text{RS}}(m_{\overline{\text{MS}}}, \nu, \nu_f)$. They are the ones expected to be of natural size. We now do not lose accuracy if we first obtain m_{RS} and later on $m_{\overline{\text{MS}}}$.

$$V_{s,\text{RS}(\text{RS}')}^{(0)}(\nu_f) = V_s^{(0)} + 2\delta m_{\text{RS}(\text{RS}')},$$

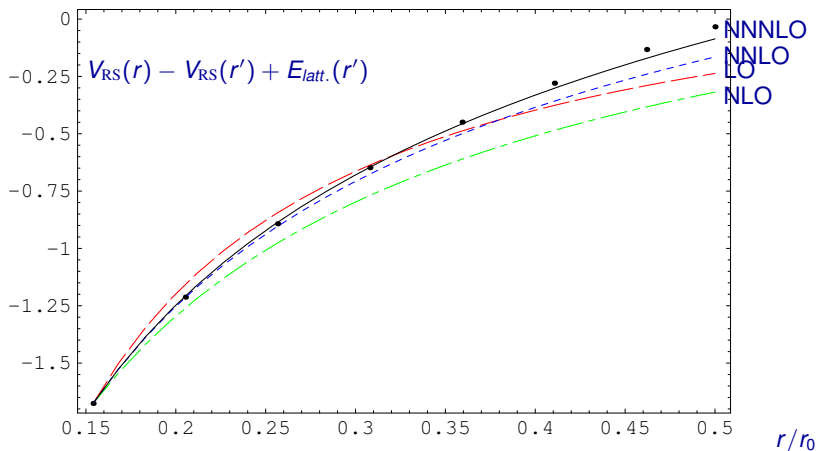


Figure: Plot of $V_{RS}(r) - V_{RS}(r') + E_{latt.}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_S(\nu)$ $\nu = \text{constant}$. $\nu_{US} = 2.5 r_0^{-1}$.

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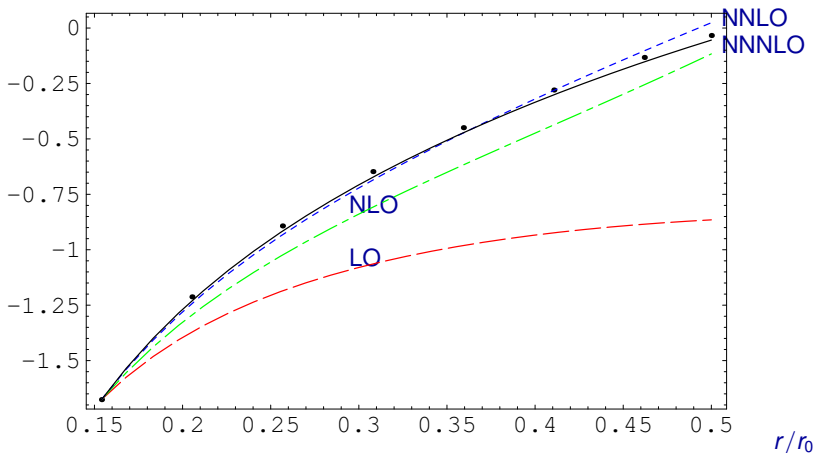


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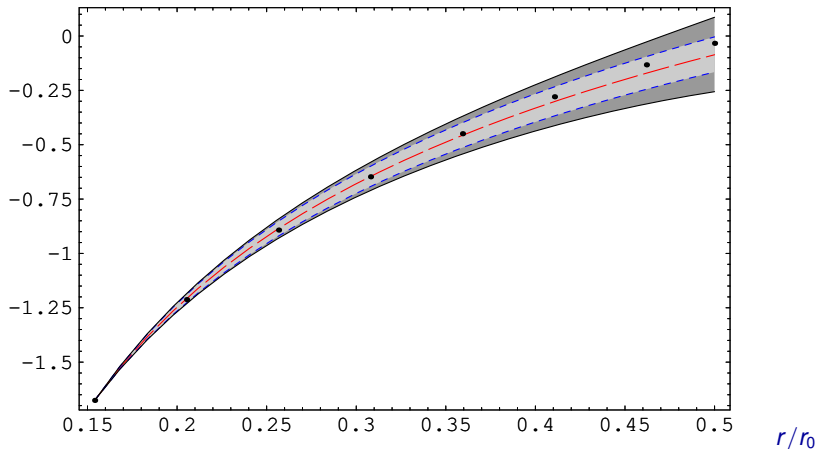


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$$V_{\text{RS}}(r) - V_{\text{RS}}(r') + E_{\text{latt.}}(r') + \sigma(r - r')$$

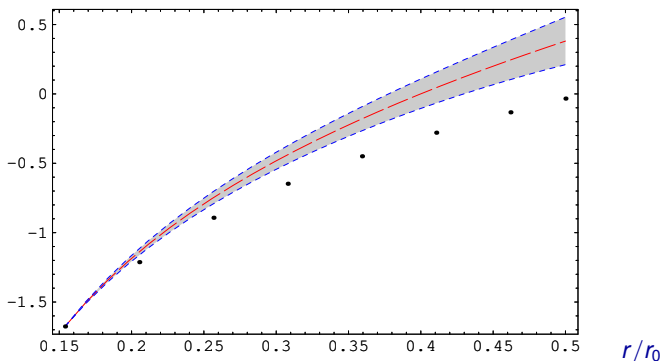


Figure: Plot of $V_{\text{RS}}(r) - V_{\text{RS}}(r') + E_{\text{latt.}}(r') + \sigma(r - r')$ versus r at three loops (estimate) with the leading ultrasoft log compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu = \text{constant}$. $\sigma = 1.35 r_0^{-2}$ and $\nu_{\text{US}} = 2.5 r_0^{-1}$.

- 1) Perturbation theory works very well for energies above 1 GeV (Pineda; Sumino, Recksiegel; Lee (Necco, Sommer for the force)).
- 2) Constraint on the size of nonperturbative effects for heavy quarkonium.
- 3) A linear non-perturbative potential at short distances is ruled out.