# LATTICE AND RENORMALONS

# IN HEAVY QUARK PHYSICS

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# Index

# Renormalons

They are a potential problem in effective field theories of QCD (OPE) where the matching coefficients can be computed in perturbation theory.

Examples:

OPE

NRQCD

HQET

pNRQCD

SCET

Renormalons appear as soon as we have factorization between different scales: They can worsen the convergence of the perturbative series in QCD.

Can one understand the renormalon within an effective field theory/factorization formalism?

Problems:

**1**) Fix the parameters of the Standard Model. Search for weakly sensitive to long distance physics observables. One wants to avoid spurious dependence on the renormalon.

**2**) Meaningful determination of non-perturbative parameters.

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n$$

Matching coefficients suffer from renormalon ambiguities that cancel with the ones of the matrix elements in effective field theory calculations.

$$c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

Its Borel transform would be

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and $c$ is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^{\infty} \mathrm{d}t \, e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the matching coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

$$\delta B[c](t) \sim \frac{1}{a-t},$$

where $a$ is a positive number, it sets up the maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation, which is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*},$$

where $n^* \sim \frac{a}{\alpha_s}$. Moreover, the fact that $a$ is positive means that, even after Borel resummation, $c$ suffers from a non-perturbative ambiguity of order

$$\delta c \sim \left(\Lambda_{QCD}\right)^{\frac{a\beta_0}{2\pi}}.$$

Examples (only true if the perturbative piece can be computed with such precision)

$$\delta_{np} M_B \sim \Lambda_{QCD}, \ \delta_{np}\Gamma(B \to X_u l\nu) \sim G_F^2 m_{\rm OS}^3 \Lambda_{QCD}^2,$$

$$\delta_{np} M_{\Upsilon(1S)} \sim m_{\rm OS} \frac{\Lambda_{QCD}^4}{(m_{\rm OS}\alpha_s)^4},$$

Problem. The OS mass suffers from renormalon ambiguities:

$$\delta_{np}^{(\rm pert.)} m_{\rm OS} = \delta_{np}^{(\rm pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{QCD}!$$

and the precision of perturbation theory is

$$\delta_{np}^{(\rm pert.)} M_B \sim \Lambda_{QCD}, \ \delta_{np}^{(\rm pert.)}\Gamma \sim G_F^2 m_{\rm OS}^4 \Lambda_{QCD},$$

$$\delta_{np}^{(\rm pert.)} M_{\Upsilon(1S)} = \delta_{np}^{(\rm pert.)} m_{\rm OS}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \cdots) \sim \Lambda_{QCD}.$$

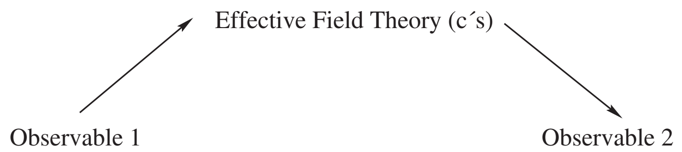



Figure 1: *Symbolic relation between observables through the determination of the matching coefficients of the effective field theory.*

Proposal: to subtract the renormalon from the matching coefficients.

Physics. Computations to n-loops produce small scales: $me^{-n}$. From the effective field theory point of view these scales should be in the effective field theory instead that in the matching coefficients.

*Future*: Factorization in dimensional regularization?

# OS mass

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}\,,$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\rm OS}](t(u)) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (\text{analytic term}),$$

**A)** $b$, $c_1$, $c_2$, ...; Renormalization group: Parisi, Beneke, Pineda. Next renormalon at $u = 1$.

$$r_n \overset{n\to\infty}{=} N_m\,\nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots \right).$$

$$b = \frac{\beta_1}{2\beta_0^2}\,, \qquad c_1 = \frac{1}{4\,b\beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right)\,, \qquad \cdots$$

**B)** Determination of $N_m$ (Pineda; Lee; Cvetic)

$$\begin{aligned} D_m(u) &= \sum_{n=0}^{\infty} D_m^{(n)} u^n = (1-2u)^{1+b} B[m_{\rm OS}](t(u)) \\ &= N_m\nu \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (1-2u)^{1+b}(\text{analytic term})\,. \end{aligned}$$

$$N_m \nu = D_m(u=1/2).$$

$$\begin{aligned} N_m &= 0.4244 + 0.1379 + 0.0127 = 0.5750 \quad (n_f = 3) \\ &= 0.4244 + 0.1275 + 0.0004 = 0.5523 \quad (n_f = 4) \\ &= 0.4244 + 0.1199 - 0.0208 = 0.5235 \quad (n_f = 5) \end{aligned}$$

$\nu \sim m$

Large $\beta_0$ analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1 + (2u-1)\ln\frac{\nu}{m} + \cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u - 1 \ll 1$)

$$(2u-1)\ln\frac{\nu}{m} \ll 1\,.$$

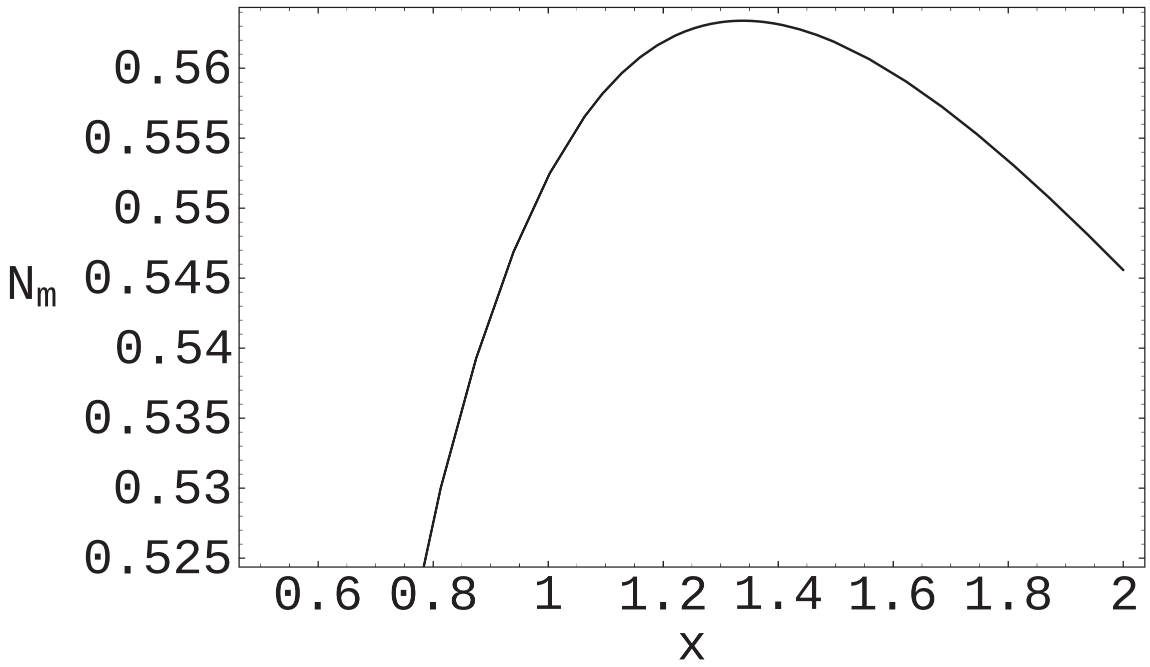



Figure 2: $x \equiv \frac{\nu}{m_{\overline{\rm MS}}}$ *dependence of* $N_m$ *for* $n_f = 4$.

# Estimates of $r_n$

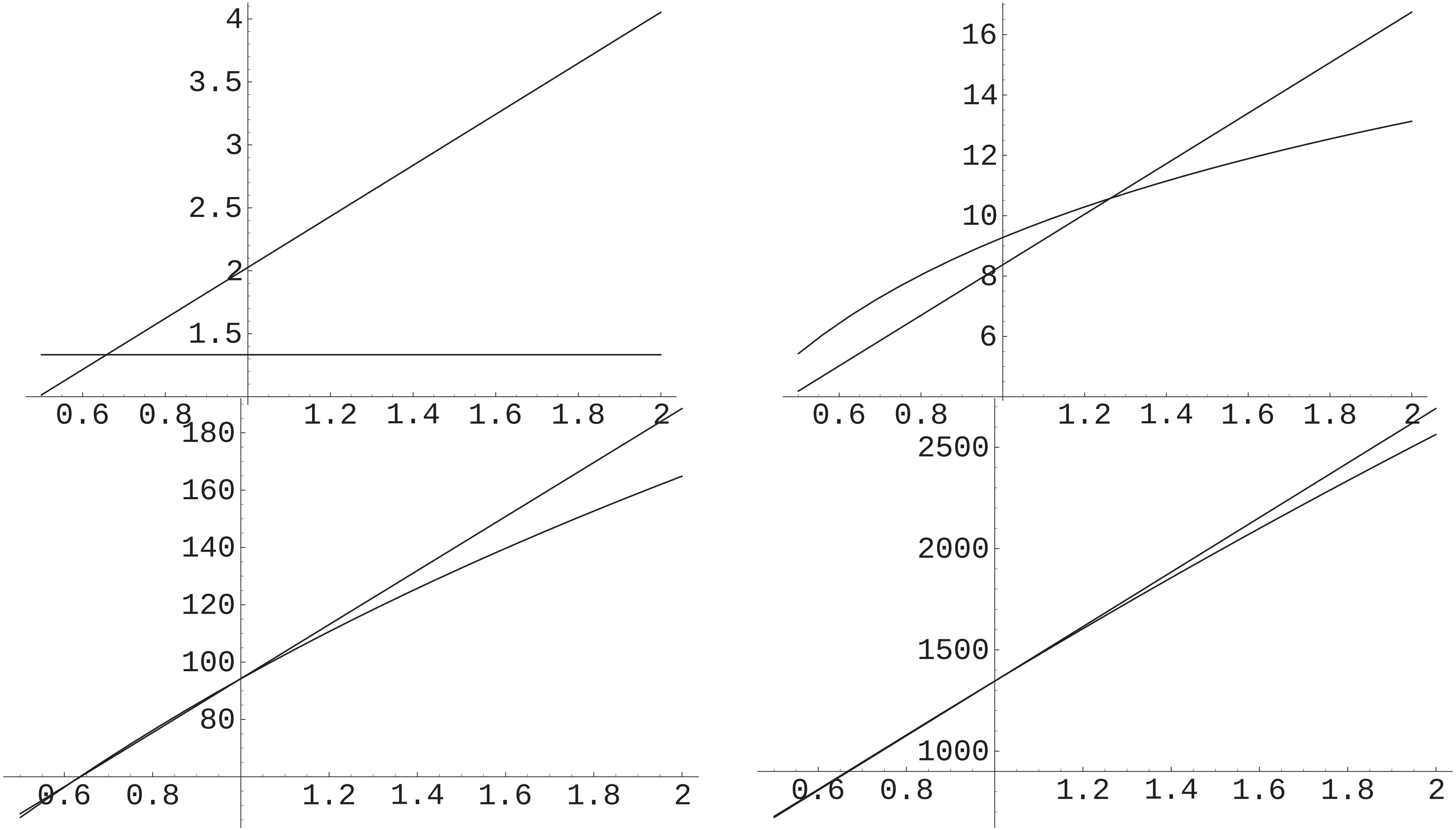
| $\tilde{r}_n = r_n/m_{\overline{\rm MS}}$ | $\tilde{r}_0$ | $\tilde{r}_1$ | $\tilde{r}_2$ | $\tilde{r}_3$ | $\tilde{r}_4$ |
|---|---|---|---|---|---|
| exact ($n_f = 3$) | 0.424413 | 1.04556 | 3.75086 | $---$ | $---$ |
| our estimate ($n_f = 3$) | 0.617148 | 0.977493 | 3.76832 | 18.6697 | 118.441 |
| large $\beta_0$ ($n_f = 3$) | 0.424413 | 1.42442 | 3.83641 | 17.1286 | 97.5872 |
| exact ($n_f = 4$) | 0.424413 | 0.940051 | 3.03854 | $---$ | $---$ |
| our estimate ($n_f = 4$) | 0.645181 | 0.848362 | 3.03913 | 13.8151 | 80.5776 |
| large $\beta_0$ ($n_f = 4$) | 0.424413 | 1.31891 | 3.28911 | 13.5972 | 71.7295 |
| exact ($n_f = 5$) | 0.424413 | 0.834538 | 2.36832 | $---$ | $---$ |
| our estimate ($n_f = 5$) | 0.706913 | 0.713994 | 2.36440 | 9.73117 | 51.5952 |
| large $\beta_0$ ($n_f = 5$) | 0.424413 | 1.21339 | 2.78390 | 10.5880 | 51.3865 |

Table 1: *Values of $r_n$ for $\nu = m_{\overline{\rm MS}}$. Either the exact result (when available), our estimate, or the estimate using the large $\beta_0$ approximation.*

| $\tilde{r}_n = r_n/m_{\overline{\rm MS}}$ | $\tilde{r}_0$ | $\tilde{r}_1$ | $\tilde{r}_2$ | $\tilde{r}_3$ | $\tilde{r}_4$ |
|---|---|---|---|---|---|
| $O(1/n)$ ($n_f = 3$) | $-0.164$ | $-0.046$ | $-0.027$ | $-0.019$ | $-0.015$ |
| $O(1/n^2)$ ($n_f = 3$) | 0.237 | $-0.103$ | $-0.017$ | $-0.007$ | $-0.004$ |
| $O(1/n)$ ($n_f = 4$) | $-0.105$ | $-0.028$ | $-0.016$ | $-0.012$ | $-0.009$ |
| $O(1/n^2)$ ($n_f = 4$) | 0.274 | $-0.126$ | $-0.020$ | $-0.008$ | $-0.004$ |
| $O(1/n)$ ($n_f = 5$) | 0.024 | 0.006 | 0.003 | 0.002 | 0.002 |
| $O(1/n^2)$ ($n_f = 5$) | 0.326 | $-0.165$ | $-0.023$ | $-0.009$ | $-0.005$ |

Table 2: $O(1/n)$ *corrections (normalized with respect the leading solution) of our $r_n$ estimates for different number of light fermions.*

$$c_1(n_f = 0) \simeq -0.215, \quad c_2(n_f = 0) \simeq 0.185$$


4
3.5
3
2.5
2
1.5
0.6 0.8 1.2 1.4 1.6 1.8 2
16
14
12
10
8
6
0.6 0.8 1.2 1.4 1.6 1.8 2
180
160
140
120
100
80
0.6 0.8 1.2 1.4 1.6 1.8 2
2500
2000
1500
1000
0.6 0.8 1.2 1.4 1.6 1.8 2

# The static potential

$$V_s^{(0)}(r;\nu_{us}) = \sum_{n=0}^{\infty} V_{s,n}^{(0)}\alpha_s^{n+1},$$

$2m_{\rm OS}+V_s^{(0)}$ (not $2m_{\rm OS}+V_o^{(0)}$) can be understood as an observable up to $O(r^2\Lambda_{QCD}^3, \Lambda_{QCD}^2/m)$ renormalon (and/or non-perturbative) contributions. We can use our knowledge of the asymptotic behavior of $m_{\rm OS}$.

$$V_{s,n}^{(0)} \overset{n\rightarrow\infty}{=} N_V\,\nu\left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right)$$

$$\boxed{2N_m + N_V = 0}$$

$$\begin{aligned} D_V(u) &= \sum_{n=0}^{\infty} D_V^{(n)}u^n = (1-2u)^{1+b}B[V_s^{(0)}](t(u)) \\ &= N_V\nu\left(1+c_1(1-2u)+c_2(1-2u)^2+\cdots\right)+(1-2u)^{1+b}(\text{analytic term}). \end{aligned}$$

Next (IR) renormalon at $u=3/2$.

$$\begin{aligned} N_V &= -1.333+0.572-0.345=-1.107 \quad (n_f=3) \\ &= -1.333+0.585-0.329=-1.077 \quad (n_f=4) \\ &= -1.333+0.587-0.295=-1.042 \quad (n_f=5)\,. \end{aligned}$$

$$2\frac{2N_m+N_V}{2N_m-N_V} = \begin{cases} 0.038 & , n_f=3 \\ 0.025 & , n_f=4 \\ 0.005 & , n_f=5. \end{cases}$$

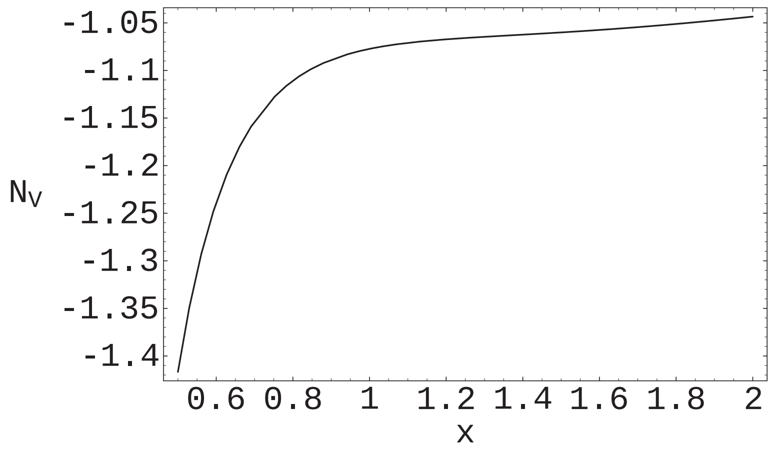



Figure 3: $x \equiv \nu r$ *dependence of* $N_V$ *for* $n_f = 4$.

| $\tilde{V}^{(0)}_{s,n} = rV^{(0)}_{s,n}$ | $\tilde{V}^{(0)}_{s,0}$ | $\tilde{V}^{(0)}_{s,1}$ | $\tilde{V}^{(0)}_{s,2}$ | $\tilde{V}^{(0)}_{s,3}$ | $\tilde{V}^{(0)}_{s,4}$ |
|---|---|---|---|---|---|
| exact ($n_f = 3$) | −1.33333 | −1.84512 | −7.28304 | − − − | − − − |
| our estimate ($n_f = 3$) | −1.23430 | −1.95499 | −7.53665 | −37.3395 | −236.882 |
| large $\beta_0$ ($n_f = 3$) | −1.33333 | −2.69395 | −7.69303 | −34.0562 | − − − |
| exact ($n_f = 4$) | −1.33333 | −1.64557 | −5.94978 | − − − | − − − |
| our estimate ($n_f = 4$) | −1.29036 | −1.69672 | −6.07826 | −27.6301 | −161.155 |
| large $\beta_0$ ($n_f = 4$) | −1.33333 | −2.49440 | −6.59553 | −27.0349 | − − − |
| exact ($n_f = 5$) | −1.33333 | −1.44602 | −4.70095 | − − − | − − − |
| our estimate ($n_f = 5$) | −1.41383 | −1.42799 | −4.72881 | −19.4623 | −103.190 |
| large $\beta_0$ ($n_f = 5$) | −1.33333 | −2.29485 | −5.58246 | −21.0518 | − − − |

Table 3: *Values of* $V^{(0)}_{s,n}$ *with* $\nu = 1/r$. *Either the exact result (when available), our estimate, or the estimate using the large* $\beta_0$ *approximation.*

# Renormalon subtracted matching and power counting

Effective field theory with renormalon free parameters but preserving the power counting rules.

The renormalon is associated to the non-analytic behavior in $1-2u$. These terms also exist in the effective theory. Procedure: to explicitly subtract them from the matching coefficients (the mass).

$$B[m_{\rm RS}] \equiv B[m_{\rm OS}] - N_m\nu_f \frac{1}{(1-2u)^{1+b}}\left(1+c_1(1-2u)+c_2(1-2u)^2+\cdots\right),$$

$$\boxed{m_{\rm RS}(\nu_f) = m_{\rm OS} - \delta m_{\rm RS} = m_{\rm OS} - \sum_{n=0}^{\infty} N_m\,\nu_f\left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f)\sum_{k=0}^{\infty} c_k\frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}\,.}$$

Expansion in $\alpha_s(\nu)$

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS}\alpha_s^{n+1}\,,$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}},\nu,\nu_f)$. They are the ones expected to be of natural size. We now do not loose accuracy if we first obtain $m_{\rm RS}$ and later on $m_{\overline{\rm MS}}$.

Different scheme

$$B[m_{RS'}] \equiv B[m_{RS}] + N_m\nu_f\left(1+c_1+c_2+\cdots\right)\,.$$

# Check of convergence improvement

| Masses | $O(\alpha_s)$ | $O(\alpha_s^2)$ | $O(\alpha_s^3)$ | $O(\alpha_s^4)$ | total |
|---|---|---|---|---|---|
| $m_{\rm OS}$ | 401 | 199 | 144 | 147 | 5 102 |
| $m_{\rm RS}$ | 111 | 50 | 17 | 7 | 4 395 |
| $m_{\rm RS'}$ | 401 | 114 | 38 | 15 | 4.778 |
| $m_{\rm PS}$ | 210 | 80 | 42 | − − − | 4 542 |
| $m_{1S}^{(\rm static)}$ | 102 | 50 | 19 | 8 | 4 389 |
| $m_{\rm RS}$ | 256 | 95 | 40 | 21 | 4 622 |
| $m_{\rm RS'}$ | 401 | 157 | 74 | 41 | 4.882 |
| $m_{\rm PS}$ | 306 | 120 | 67 | − − − | 4.703 |
| $m_{1S}^{(\rm static)}$ | 251 | 94 | 41 | 22 | 4 619 |

Table 4: *Contributions at various orders in $\alpha_s$ for different mass definitions for the bottom quark case, either with $\nu_f = 1/r = 2$ GeV (middle panel) or with $\nu_f = 1/r = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 1 has been used. The other parameters have been fixed to the values $m_{\overline{\rm MS}}(m_{\overline{\rm MS}}) = 4.21$ GeV, $\nu = m_{\overline{\rm MS}}(m_{\overline{\rm MS}})$ and $n_f = 4$.*

$$m_{1S}^{(\rm static)} \equiv m_{\rm OS} + \frac{V(r)}{2} = m_{\overline{\rm MS}} + \left( r_0 - \frac{C_f}{2r} \right) \alpha_s + \cdots .$$

# HQET

$$\mathcal{L} = \bar{h}\,(iD_0 - \delta m_{\rm RS})\,h + O\left(\frac{1}{m_{RS}}\right)\,,$$

where $\delta m_{\rm RS} = m_{\rm OS} - m_{\rm RS}$ and similarly for the NRQCD Lagrangian.

Weakly sensitive to long distance physics observable

$$\langle M_B \rangle - \langle M_D \rangle = m_{b,\rm RS} - m_{c,\rm RS} + \lambda_1 \left(\frac{1}{2m_{b,\rm RS}} - \frac{1}{2m_{c,\rm RS}}\right) + O(1/m_{\rm RS}^2)\,.$$

$$\bar{\Lambda}_{\rm RS} = \langle M_B \rangle - m_{b,\rm RS} - \frac{\lambda_1}{2m_{b,\rm RS}} + O(1/m_{b,\rm RS}^2)\,.$$

# pNRQCD. If $\Lambda_{QCD} \ll m\alpha_s$

$$V^{(0)}_{s,\mathrm{RS(RS')}}(\nu_f) = V^{(0)}_s + 2\delta m_{\mathrm{RS(RS')}} ,$$

## Check of convergence improvement

| Potentials | $O(\alpha_s)$ | $O(\alpha_s^2)$ | $O(\alpha_s^3)$ | $O(\alpha_s^4)$ | total |
|---|---|---|---|---|---|
| $V_s^{(0)}$ | $-910$ | $-306$ | $-302$ | $-383$ | $-1\,902$ |
| $V_{s,\mathrm{RS}}^{(0)}$ | $-205$ | 3 | $-2$ | $-3$ | $-208$ |
| $V_{s,\mathrm{RS'}}^{(0)}$ | $-910$ | $-54$ | $-14$ | $-6$ | $-984$ |
| $V_{s,\mathrm{PS}}^{(0)}$ | $-446$ | $-42$ | $-25$ | $---$ | $-513$ |
| $V_{s,\mathrm{RS}}^{(0)}$ | $-558$ | $-63$ | $-41$ | $-26$ | $-687$ |
| $V_{s,\mathrm{RS'}}^{(0)}$ | $-910$ | $-180$ | $-95$ | $-54$ | $-1\,239$ |
| $V_{s,\mathrm{PS}}^{(0)}$ | $-678$ | $-116$ | $-75$ | $---$ | $-869$ |

Table 5: *Contributions at various orders in $\alpha_s$ for different singlet static potential definitions for some typical scales in the $\Upsilon$ system, either with $\nu_f = 2$ GeV (middle panel) or with $\nu_f = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 3 has been used. The other parameters have been fixed to the values $\nu = 1/r = 2.5$ GeV and $n_f = 4$.*

## pNRQCD Lagrangian

$$\mathcal{L}^{(0)} = \mathrm{Tr}\Big\{S^\dagger\left(i\partial_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{s,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right)S + O^\dagger\left(iD_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{o,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right)O\Big\}$$
$$+\; gV_A(\mathbf{x})\mathrm{Tr}\left\{O^\dagger\mathbf{x}\cdot\mathbf{E}\,S + S^\dagger\mathbf{x}\cdot\mathbf{E}\,O\right\} + g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{O^\dagger\mathbf{x}\cdot\mathbf{E}\,O + O^\dagger O\mathbf{x}\cdot\mathbf{E}\right\},$$

## Weakly sensitive to long distance physics observable

$$M_{nlj} = 2m_{\mathrm{RS}} + \sum_{m=2}^{\infty} A^{m,\mathrm{RS}}_{nlj}(\nu_{us})\alpha_s^m + \delta M^{\mathrm{US}}_{nlj}(\nu_{us}) .$$

# pNRQCD: the static limit $O(1/m^0)$

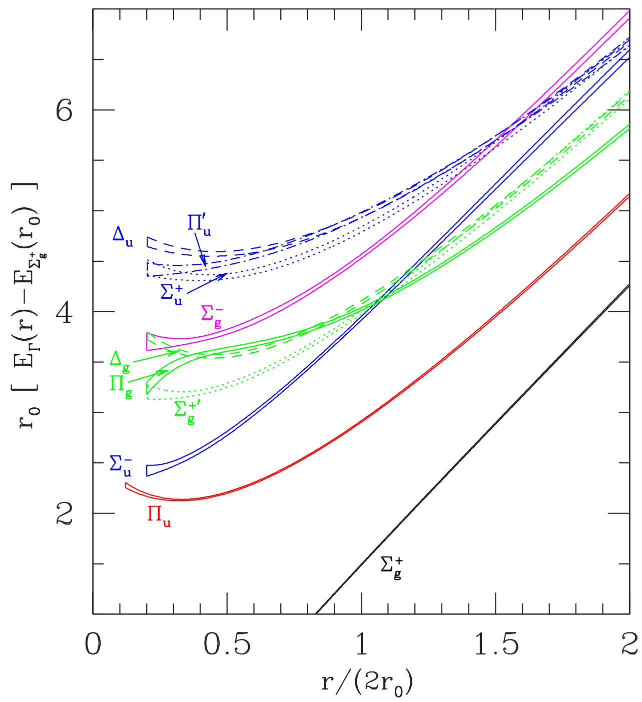



Figure 4: *Energies for different gluonic excitation between static quarks. Juge et al. hep-lat/9809015,* $r_0 \simeq 0.5$ *fm.*

$$E_s = 2m_{\rm RS}(\nu_f) + V_{s,{\rm RS}}(\nu_f) + \mathcal{O}(r^2) \qquad V_s = -C_F\alpha_s/r + \cdots$$

$$E_H = 2m_{\rm RS}(\nu_f) + V_{o,{\rm RS}}(\nu_f) + \Lambda_H^{\rm RS}(\nu_f) + \mathcal{O}(r^2) \qquad V_o = 1/(2N_c)\alpha_s/r + \cdots$$

# The static singlet potential

Pineda

The introduction of renormalons allows to obtain agreement between lattice simulations and perturbation theory.

$$E_s = 2m_{\rm OS} + V_{s,\rm OS} + \mathcal{O}(r^2)$$

$$E_s = 2m_{\rm RS}(\nu_f) + V_{s,\rm RS}(\nu_f) + \mathcal{O}(r^2)$$

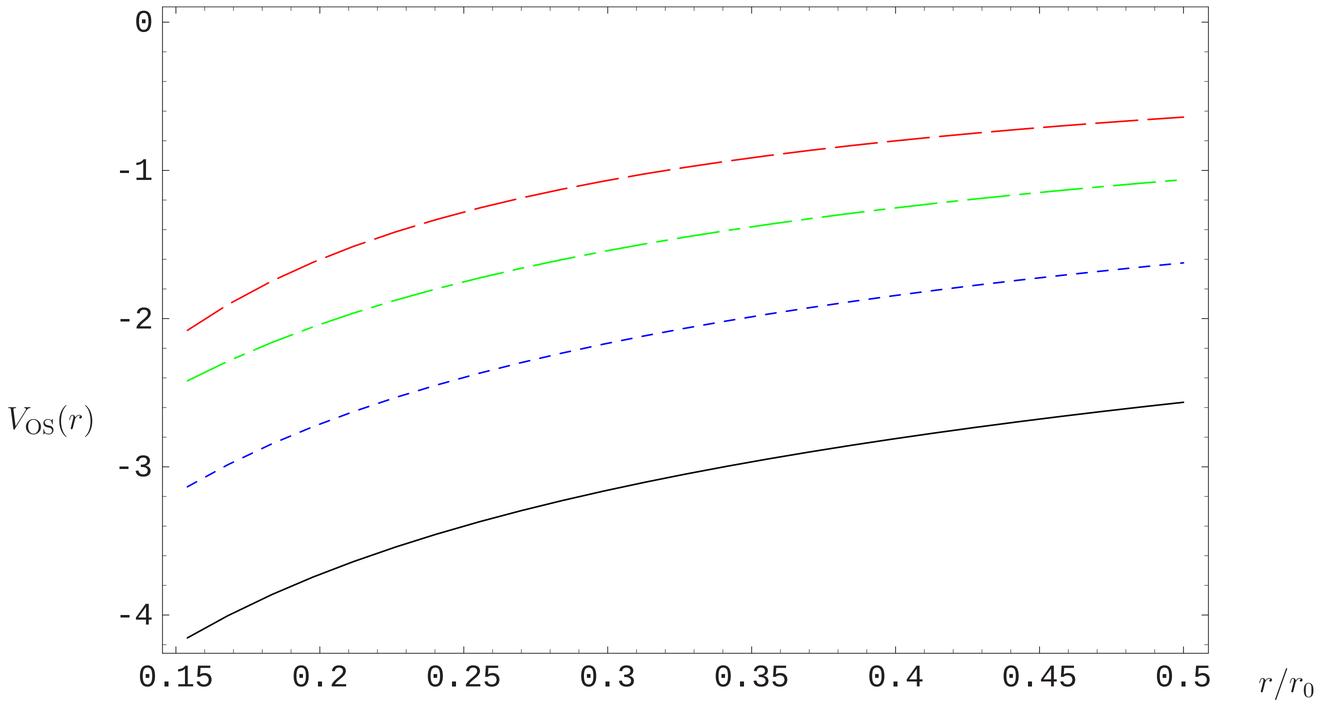



Figure 5: *Plot of $V_{\rm OS}(r)$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line). For the scale of $\alpha_s(\nu)$ $\nu$ = constant. $\nu_{us} = 2.5\, r_0^{-1}$.*

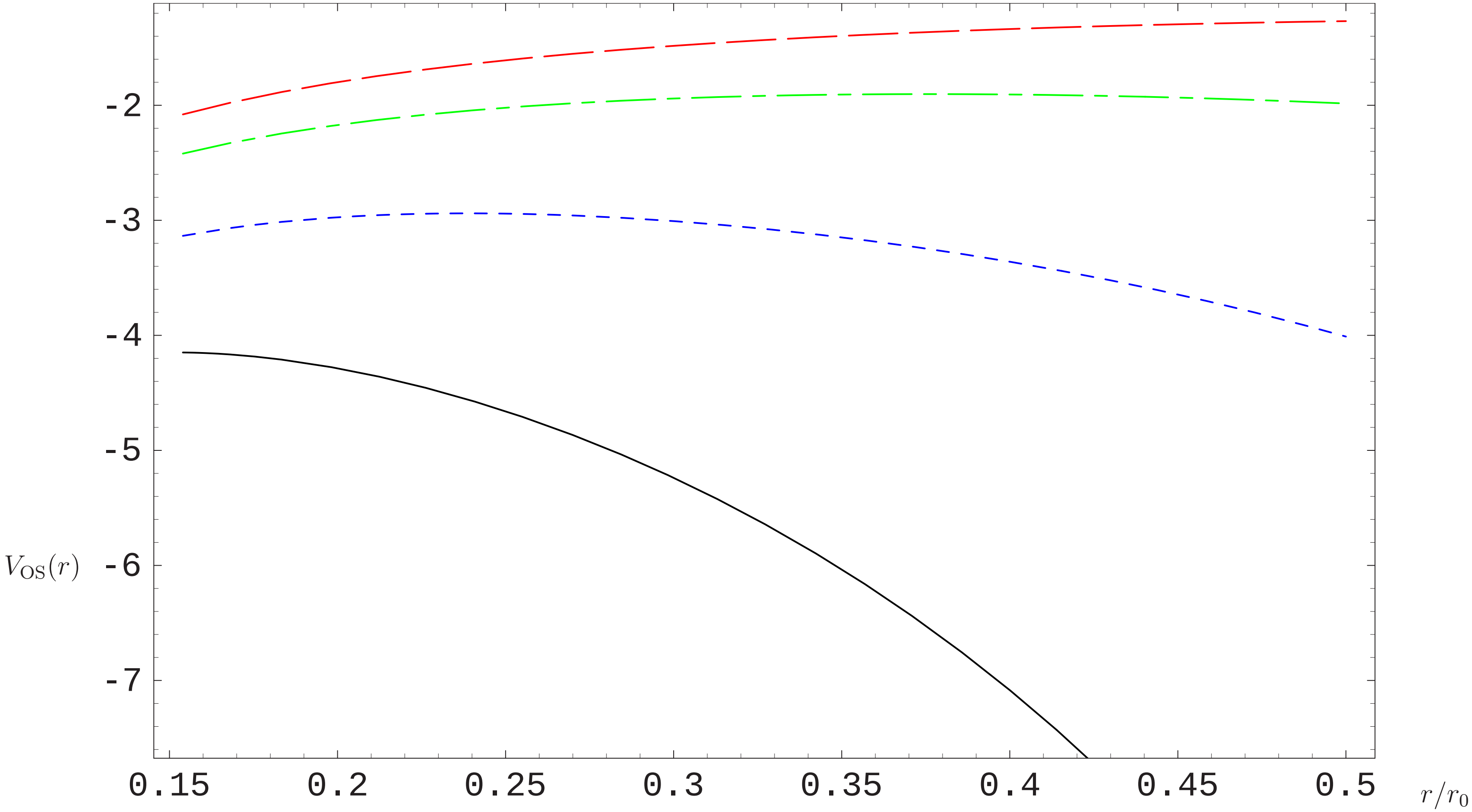



Figure 6: *Plot of $V_{\rm OS}(r)$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line). For the scale of $\alpha_s(\nu)$, we set $\nu = 1/r$. $\nu_{us} = 2.5\, r_0^{-1}$.*

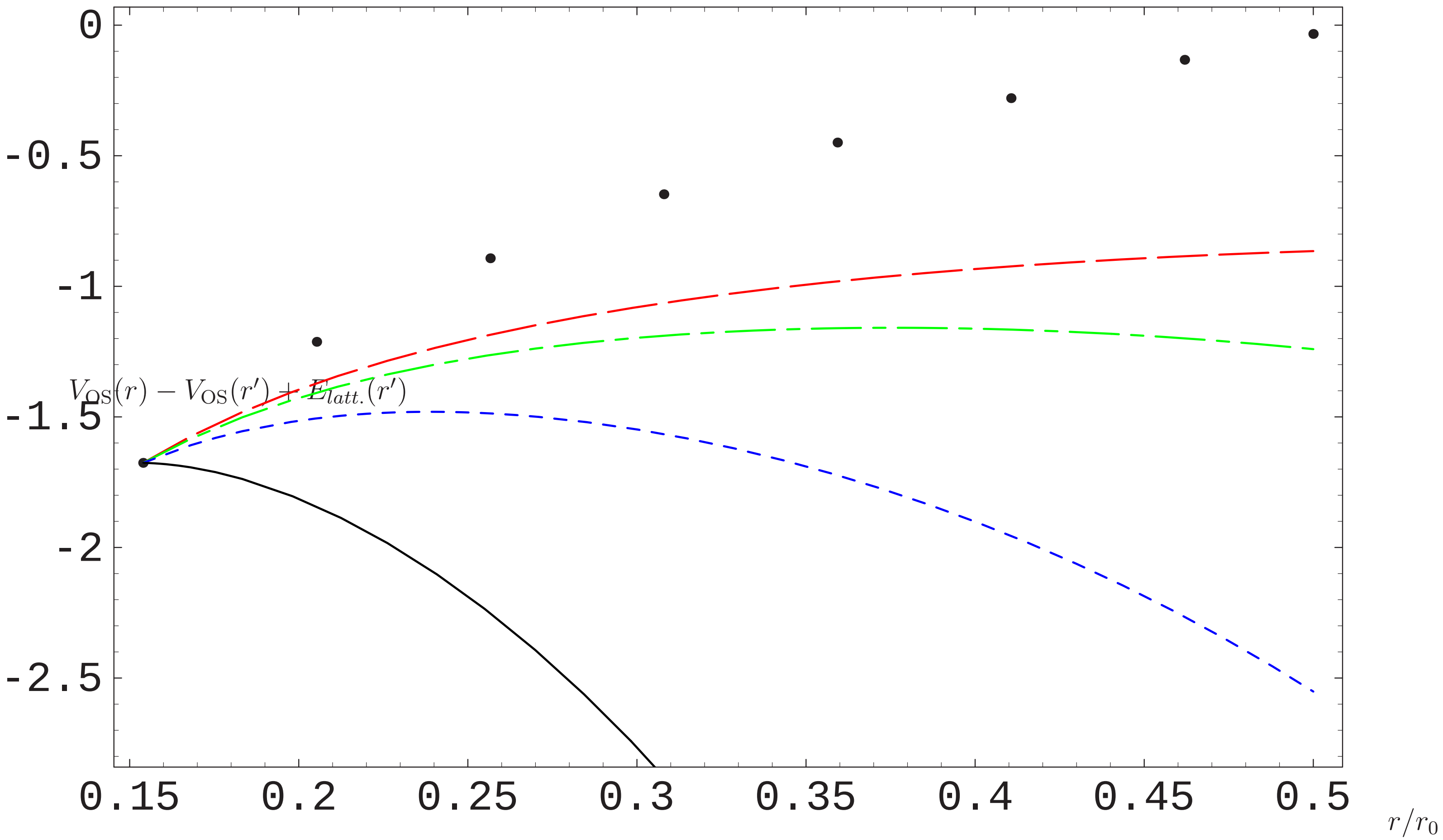



Figure 7: *Plot of $V_{\rm OS}(r) - V_{\rm OS}(r') + E_{latt.}(r')$ versus $r$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu = 1/r$. $\nu_{us} = 2.5\, r_0^{-1}$.*

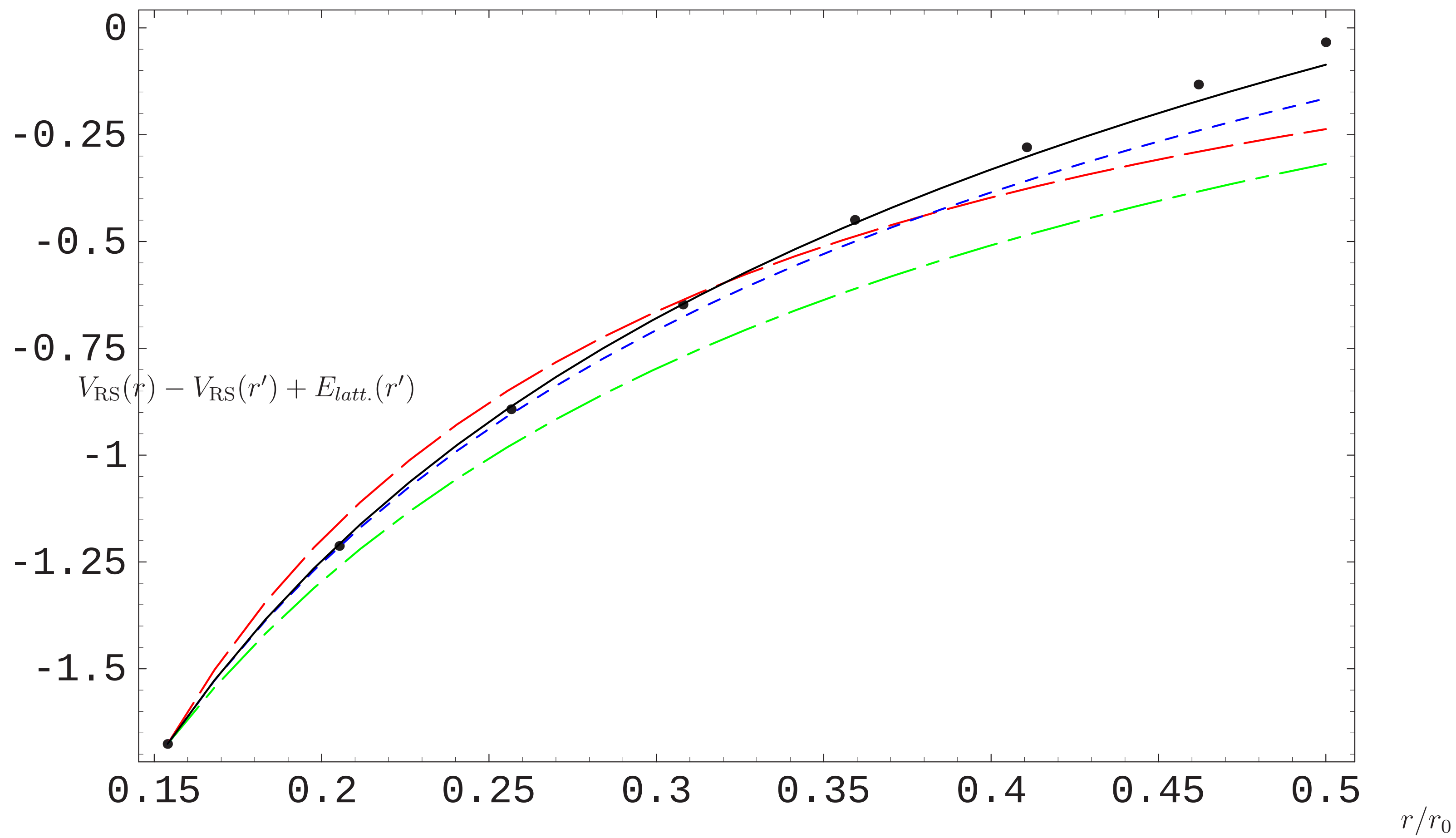



Figure 8: *Plot of $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ versus $r$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$ $\nu = constant$. $\nu_{us} = 2.5\, r_0^{-1}$.*

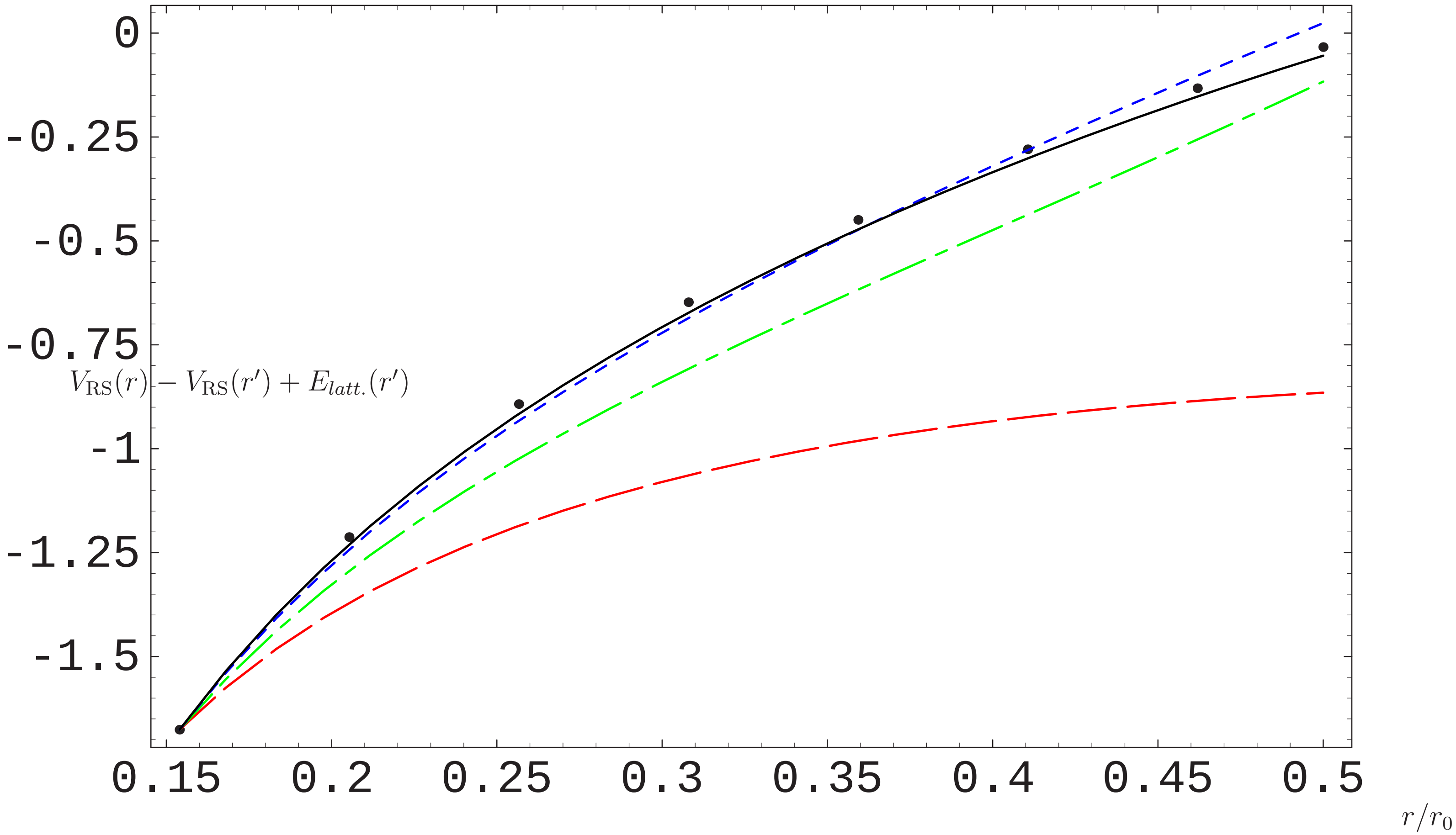



Figure 9: *Plot of* $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ *versus* $r$ *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$*, we set* $\nu = 1/r$. $\nu_f = \nu_{us} = 2.5\, r_0^{-1}$.

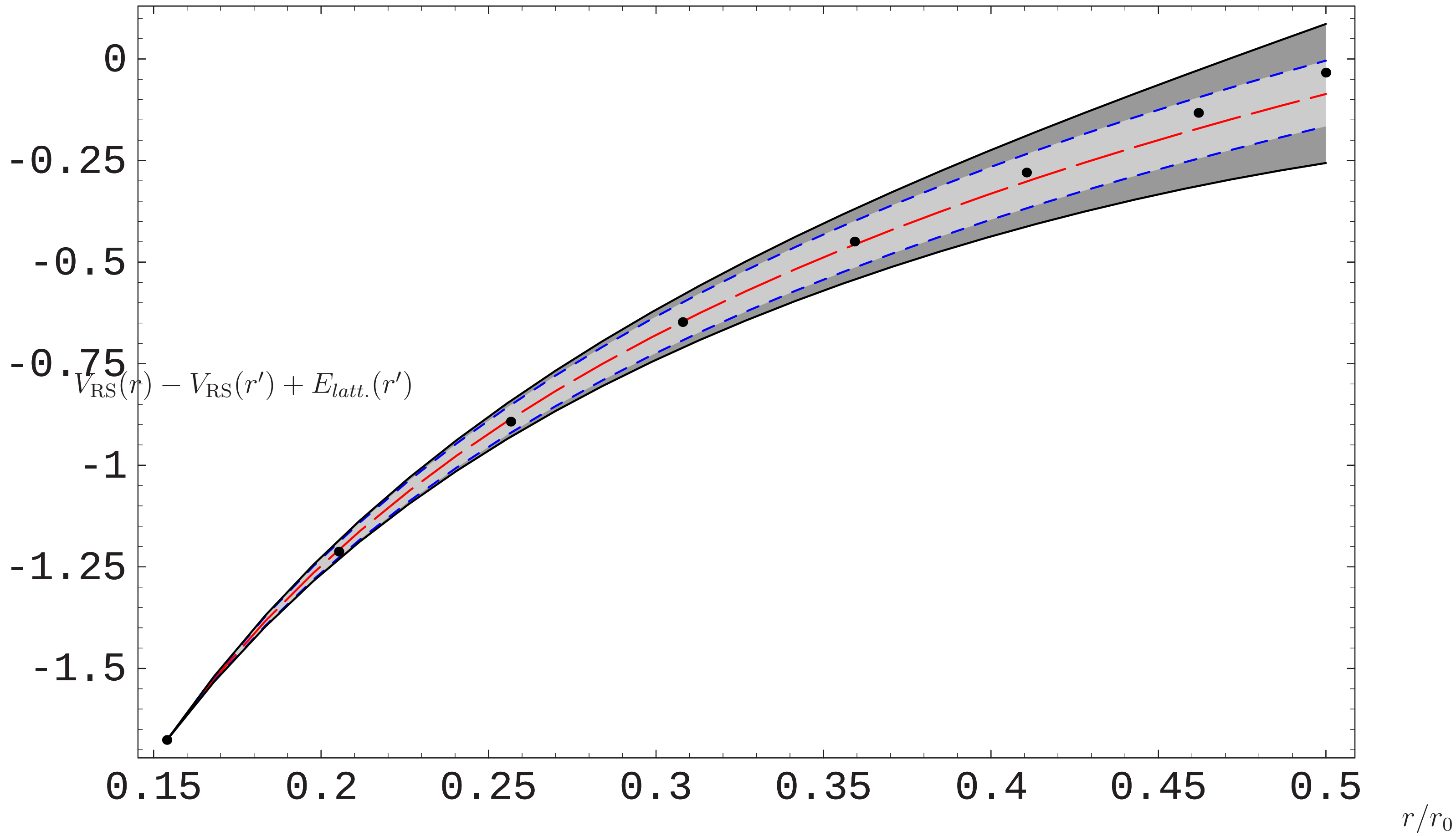



Figure 10: *Plot of* $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ *versus* $r$ *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$ $\nu = constant$. $\nu_{us} = 2.5\, r_0^{-1}$.

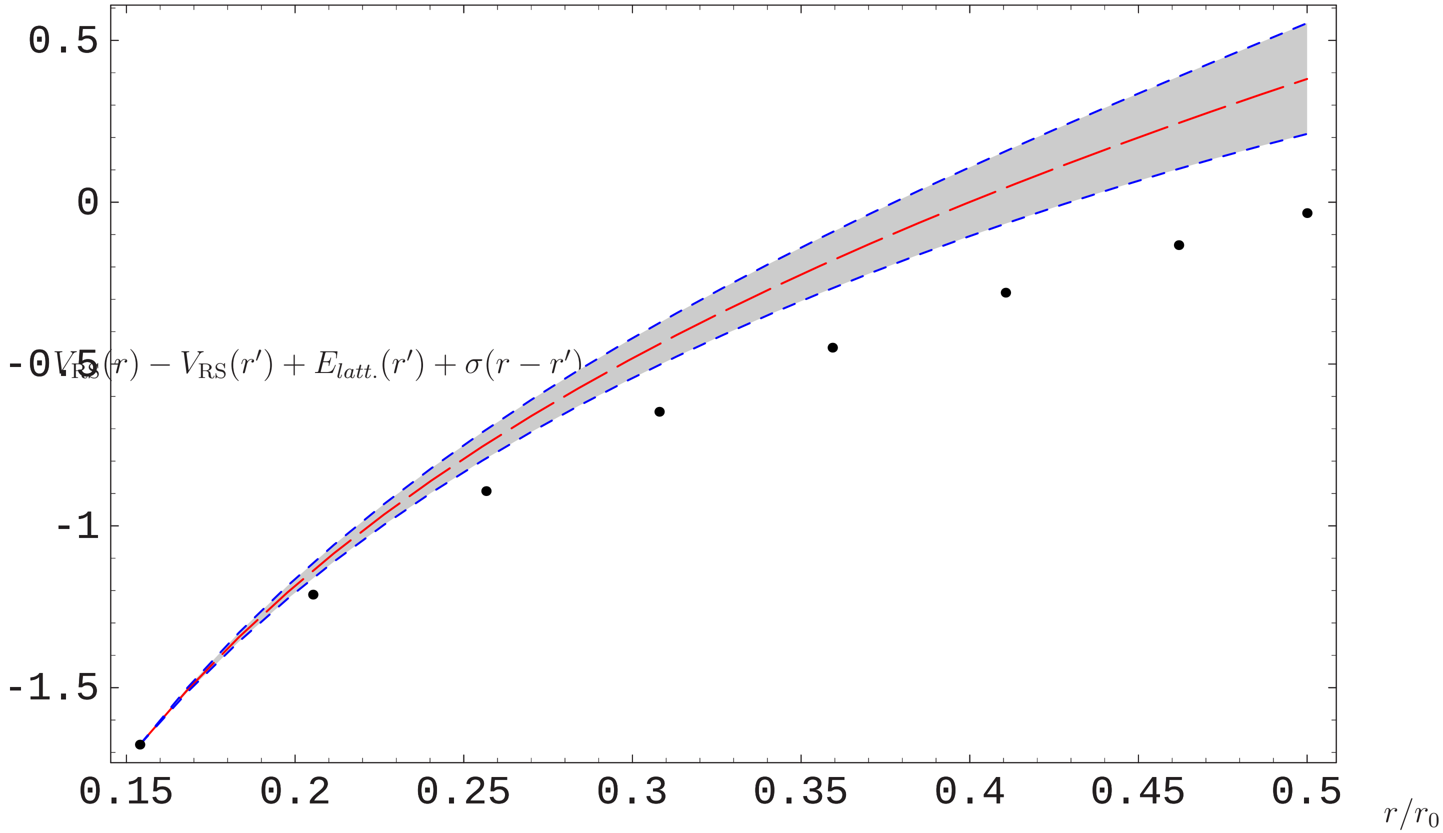



Figure 11: *Plot of $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r') + \sigma(r-r')$ versus $r$ at three loops (estimate) with the leading ultrasoft log compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu$ = constant. $\sigma = 1.35\, r_0^{-2}$ and $\nu_{us} = 2.5\, r_0^{-1}$.*

1) Perturbation theory works very well for energies above 1 GeV (Pineda; Sumino, Recksiegel; Lee).

2) Constraint on the size of nonperturbative effecs for heavy quarkonium.

3) A linear non-perturbative potential at short distances is ruled out.

# Static Hybrids versus gluelumps

Brambilla, AP, Soto, Vairo

Extra (approximate) symmetries in the effective theory.

NRQCD: $D_{\infty h}$ (substituting parity by CP).

pNRQCD: $O(3)\times$C. Softly broken by the multipole expansion.

Degeneracies:

$$\Sigma_g^{+\,\prime} \sim \Pi_g \; ; \qquad \Sigma_g^- \sim \Pi_g' \sim \Delta_g \; ;$$
$$\Sigma_u^- \sim \Pi_u \; ; \qquad \Sigma_u^+ \sim \Pi_u' \sim \Delta_u \; .$$

| Gluelumps $O^a H^a$ | $L=1$ | $L=2$ |
|---|---|---|
| $\Sigma_g^{+\,\prime}$ | $\mathbf{r}\cdot\mathbf{E}\,,\mathbf{r}\cdot(\mathbf{D}\times\mathbf{B})$ | |
| $\Sigma_g^-$ | | $(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{B})$ |
| $\Pi_g$ | $\mathbf{r}\times\mathbf{E}\,,\mathbf{r}\times(\mathbf{D}\times\mathbf{B})$ | |
| $\Pi_g'$ | | $\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{B}+\mathbf{D}(\mathbf{r}\cdot\mathbf{B}))$ |
| $\Delta_g$ | | $(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{B})^j+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{B})^i$ |
| $\Sigma_u^+$ | | $(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{E})$ |
| $\Sigma_u^-$ | $\mathbf{r}\cdot\mathbf{B}\,,\mathbf{r}\cdot(\mathbf{D}\times\mathbf{E})$ | |
| $\Pi_u$ | $\mathbf{r}\times\mathbf{B}\,,\mathbf{r}\times(\mathbf{D}\times\mathbf{E})$ | |
| $\Pi_u'$ | | $\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{E}+\mathbf{D}(\mathbf{r}\cdot\mathbf{E}))$ |
| $\Delta_u$ | | $(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{E})^j+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{E})^i$ |

Table 6: *Operators $H$ for the $\Sigma$, $\Pi$ and $\Delta$ gluonic excitations between static quarks in pNRQCD up to dimensions 3. The covariant derivative is understood in the adjoint representation. $\mathbf{D}\cdot\mathbf{B}$ and $\mathbf{D}\cdot\mathbf{E}$ do not appear, the first because it is identically zero after using the Jacobi identity, while the second gives vanishing contributions after using the equations of motion.*

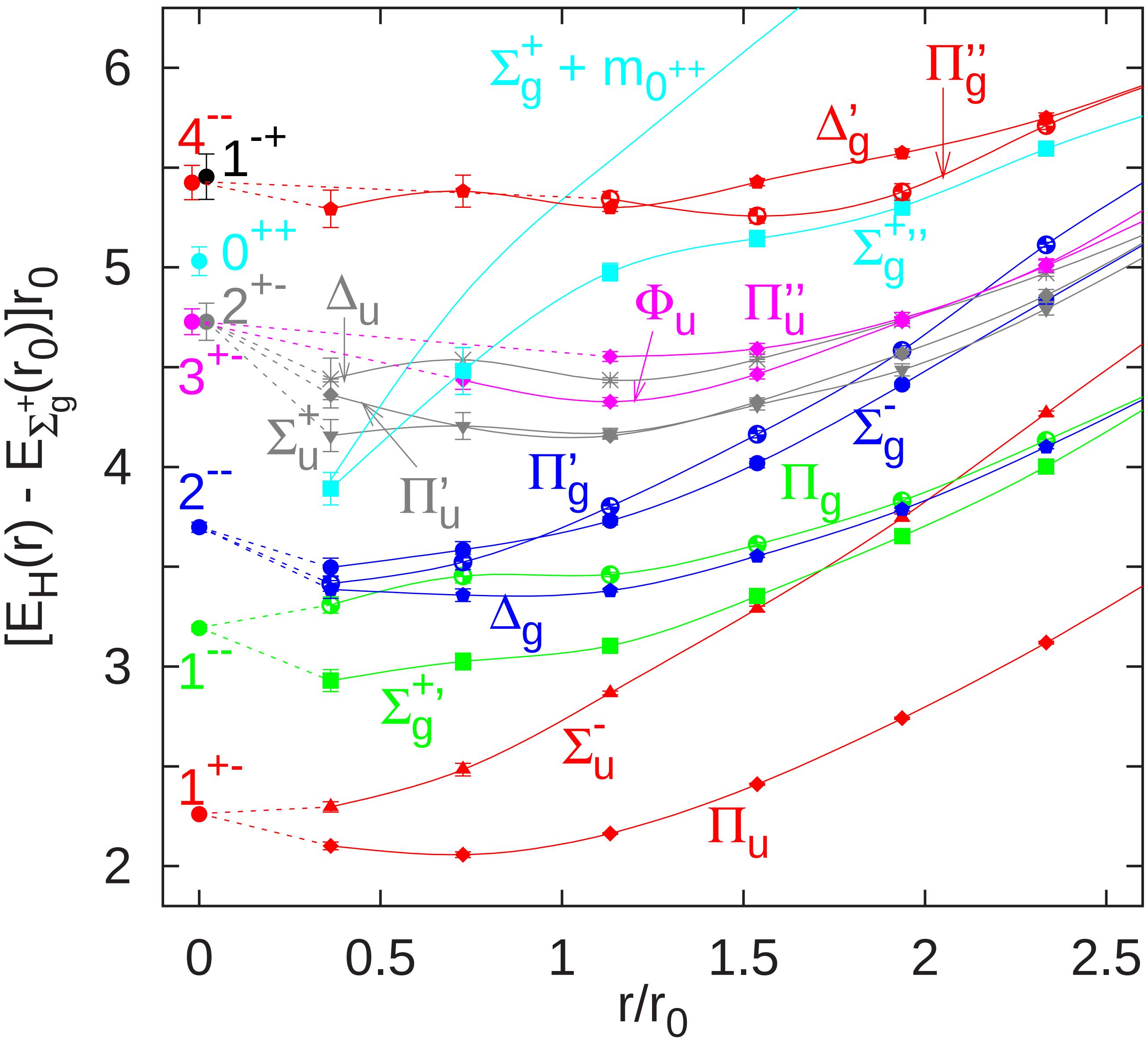



Figure 12: Different hybrid potentials Juge et al. 2003 at a lattice spacing $a_\sigma \approx 0.2$ fm $\approx 0.4\, r_0$, where $r_0 \approx 0.5$ fm, in comparison with the gluelump spectrum, extrapolated to the continuum limit Foster and Michael, 1999 (circles, left-most data points). The gluelump spectrum has been shifted by an arbitrary constant to adjust the $1^{+-}$ state with the $\Pi_u$ and $\Sigma_u^-$ potentials at short distance. In addition, we include the sum of the ground state ($\Sigma_g^+$) potential and the scalar glueball mass $m_{0^{++}}$ Bali et al., 1993; Lucini and Teper, 2001; Morningstar and Peardon, 1999. The lines are drawn to guide the eye. Bali-Pineda 2004.

## Predictions of the shape of the static energies:

We compute ($H = O^a H^a$)

$$\langle 0|H(\mathbf{R},\mathbf{r},T/2)H^\dagger(\mathbf{R}',\mathbf{r}',-T/2)|0\rangle$$
$$\sim \delta^3(\mathbf{R}-\mathbf{R}')\delta^3(\mathbf{r}-\mathbf{r}')\, e^{-iTV_H(r)}$$

for large $T$. At leading order in the multipole expansion we obtain

$$V_H(r) = V_o(r) + \frac{i}{T}\ln\langle H^a(T/2)\phi(T/2,-T/2)^{\rm adj}_{ab}H^b(-T/2)\rangle,$$

where the $T \to \infty$ limit is understood. The general structure of the gluonic correlator is the following

$$\langle H^a(T/2)\phi(T/2,-T/2)^{\rm adj}_{ab}H^b(-T/2)\rangle^{\rm nonpert.}$$
$$\simeq h\, e^{-i\Lambda_H T} + h'\, e^{-i\Lambda'_H T} + \dots .$$

Since we are in the static limit, $1/T \ll \Lambda_{\rm QCD} \sim \Lambda_{\rm H} < \Lambda'_{\rm H} < \dots$, one can approximate the right-hand side by just keeping the first exponential. Then we get at leading order in the multipole expansion

$$\boxed{V_H(r) = V_o(r) + \Lambda_H}$$

We can relate, in the short-distance limit, the behavior of the energies for the gluonic excitations between static quarks with the large time behavior of some gluonic correlators, in particular with their correlation length.

$$\langle 0|F^a_{\mu\nu}(t)\phi(t,0)^{\rm adj}_{ab}F^b_{\mu\nu}(0)|0\rangle\,.$$

One can parameterize this correlator as a function of two scalar functions:

$$\langle 0|\mathbf{E}^a(t)\phi(t,0)^{\rm adj}_{ab}\mathbf{E}^b(0)|0\rangle \quad \text{and} \quad \langle 0|\mathbf{B}^a(t)\phi(t,0)^{\rm adj}_{ab}\mathbf{B}^b(0)|0\rangle\,,$$

with correlation lengths: $T_E = 1/\Lambda_E$ and $T_B = 1/\Lambda_B$ respectively. We can conclude that

$$\boxed{T_E < T_B}$$

One could eventually go beyond by computing corrections in the multipole expansion.

$$E_H = 2m_{\rm OS} + V_{o,{\rm OS}} + \Lambda_H^{\rm OS} + \mathcal{O}(r^2)$$

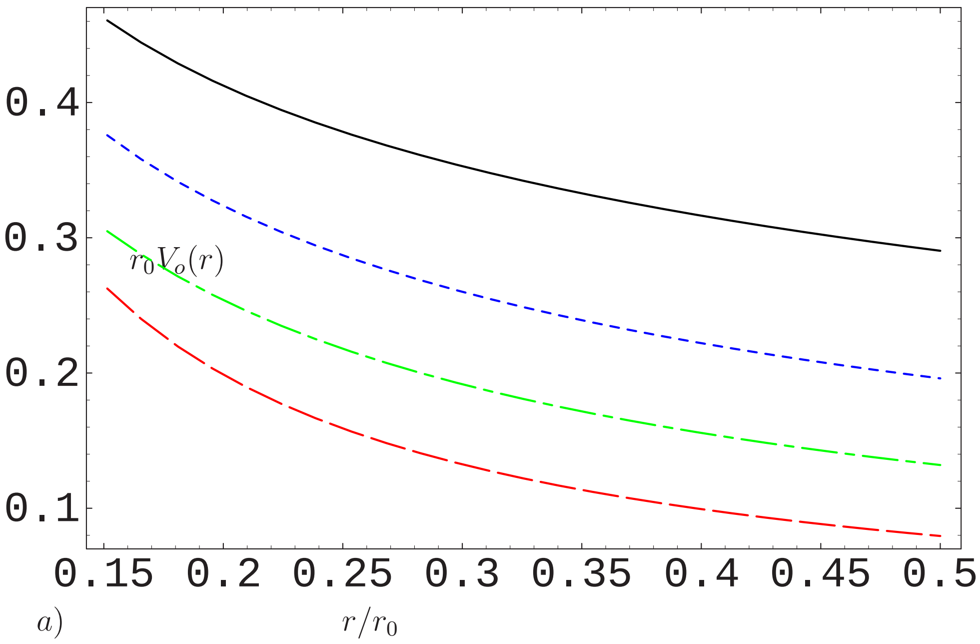
$$E_H = 2m_{\rm RS}(\nu_f) + V_{o,{\rm RS}}(\nu_f) + \Lambda_H^{\rm RS}(\nu_f) + \mathcal{O}(r^2)$$

$$E_H = 2m_{\rm latt}(1/a) + V_{o,{\rm latt}}(1/a) + \Lambda_H^{\rm latt}(1/a) + \mathcal{O}(r^2)$$

$$V_o = \frac{1}{2N_c}\frac{\alpha_s}{r} + \cdots ({\rm two-loops: Kniehl, Penin, Schroeder, Smirnov, Steinhauser})$$

$$2N_m + N_{V_o} + N_\Lambda = 0$$

Factorization scale $\nu_f \leftrightarrow 1/a$; $\delta m_{\rm RS}(\nu_f)$, $\delta\Lambda_{\rm RS}(\nu_f)$; $\delta m_{\rm latt}(1/a)$, $\delta\Lambda_{\rm latt}(1/a)$.



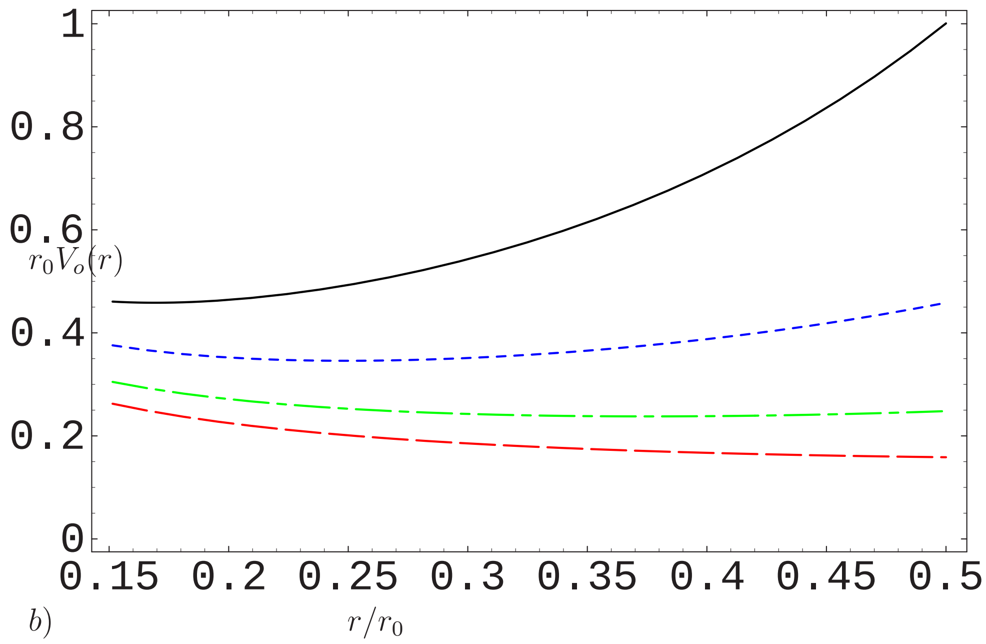




Figure 13: $r_0V_o(r)$ *(the octet potential in the OS scheme) at tree level (dashed lines), one-loop (dashed-dotted lines), two loops (dotted lines) and three loops (estimate) plus the leading single ultrasoft log (solid lines). Fig. a) corresponds to the scale* $\nu = \nu_i$ *and Fig. b) to* $\nu = 1/r$*. In both cases,* $\nu_{us} = 2.5\, r_0^{-1}$*. Only the solid curves depend on this choice.*

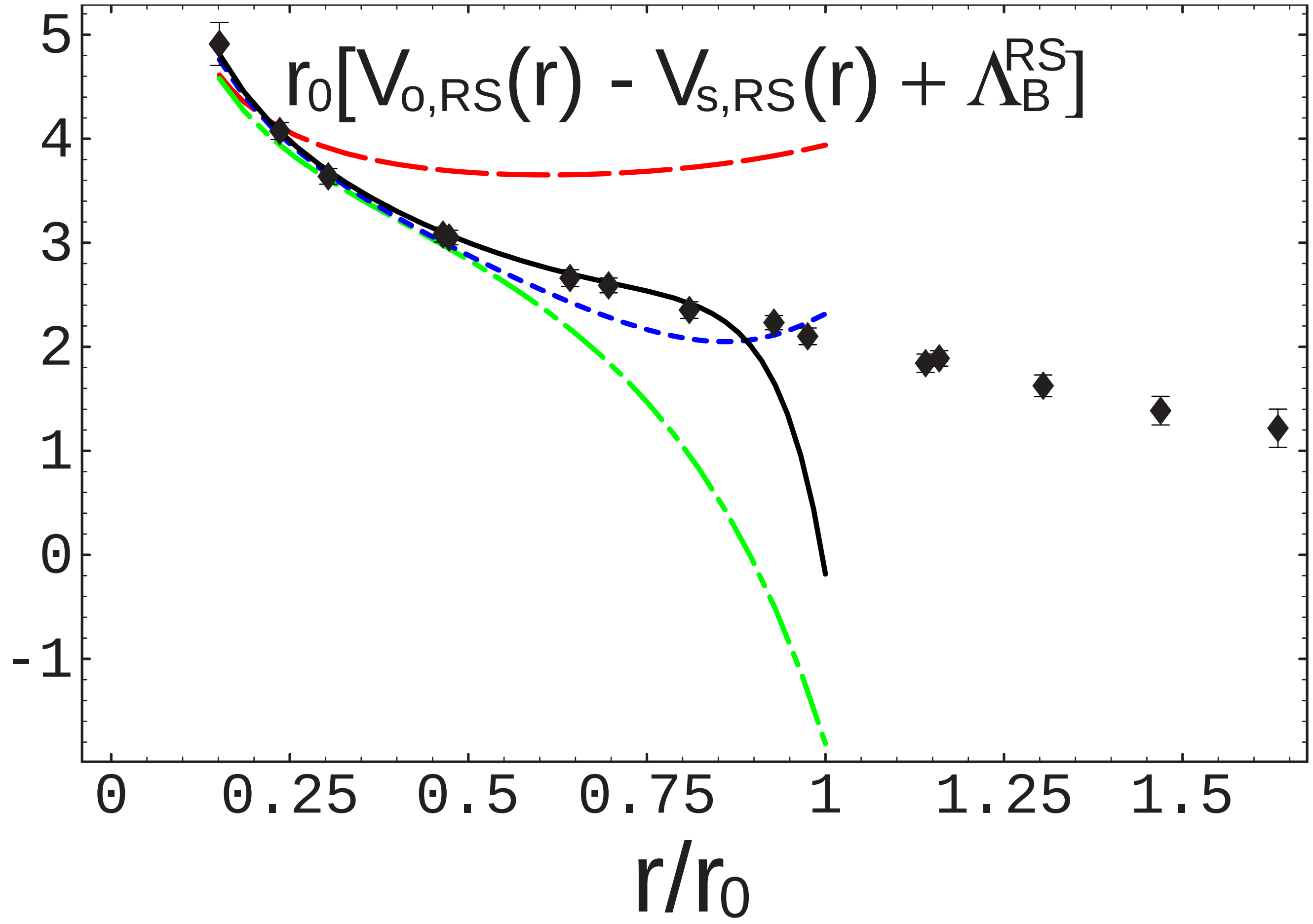



Figure 14: Splitting between the $\Pi_u$ and the $\Sigma_g^+$ potentials and the comparison with the theoretical prediction with $\nu = 1/r$ for $\nu_f = 2.5\, r_0^{-1}$. $r_0[(V_{o,\rm RS} - V_{s,\rm RS})(r) + \Lambda_B^{\rm RS}]$ is plotted versus r at tree level (dashed line), one-loop (dashed-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line). Bali-Pineda 2004.

Table 7: Absolute values for the gluelump masses in the continuum limit in the RS scheme at $\nu_f = 2.5\, r_0^{-1} \approx 1$ GeV, in $r_0$ units and in GeV. Note that an additional uncertainty of about 10 % should be added to the last column to account for the quenched approximation. We also display examples of creation operators $H$ for these states. The curly braces denote complete symmetrization of the indices. Bali-Pineda 2004.

| $J^{PC}$ | $H$ | $\Lambda_H^{\rm RS} r_0$ | $\Lambda_H^{\rm RS}/\mathrm{GeV}$ |
|---|---|---|---|
| $1^{+-}$ | $B_i$ | 2.25(39) | 0.87(15) |
| $1^{--}$ | $E_i$ | 3.18(41) | 1.25(16) |
| $2^{--}$ | $D_{\{i}B_{j\}}$ | 3.69(42) | 1.45(17) |
| $2^{+-}$ | $D_{\{i}E_{j\}}$ | 4.72(48) | 1.86(19) |
| $3^{+-}$ | $D_{\{i}D_jB_{k\}}$ | 4.72(45) | 1.86(18) |
| $0^{++}$ | $\mathbf{B}^2$ | 5.02(46) | 1.98(18) |
| $4^{--}$ | $D_{\{i}D_jD_kB_{l\}}$ | 5.41(46) | 2.13(18) |
| $1^{-+}$ | $(\mathbf{B}\wedge\mathbf{E})_i$ | 5.45(51) | 2.15(20) |

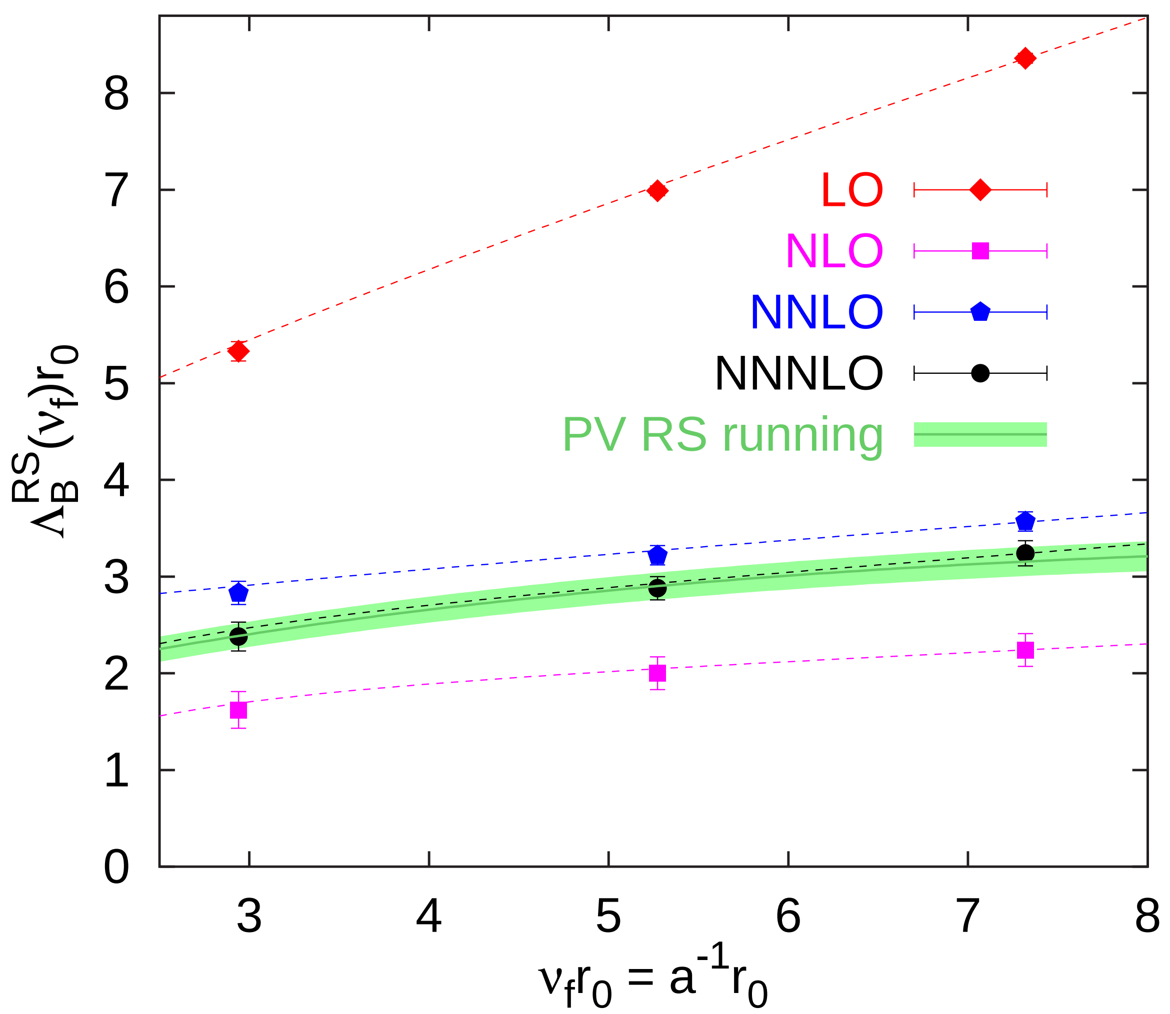



Figure 15: *The lowest gluelump mass $\Lambda_B^L$ as obtained on the lattice (diamonds), as well as converted into the RS scheme at NLO (squares), NNLO (pentagons) and NNNLO* (NNNLO estimate, circles). The error band corresponds to the result for $\Lambda_B^{RS}$ without the "theoretical" error, run to different scales, according to the PV prescription. Bali-Pineda 2004.*

# Heavy-light systems

$$M_B = m_{\rm RS}(\nu_f) + \bar{\Lambda}^{\rm RS}(\nu_f) + \mathcal{O}(1/m)$$

$$M_B = m_{\rm latt}(1/a) + \bar{\Lambda}^{\rm latt}(1/a) + \mathcal{O}(1/m)$$

$$\boxed{\overline{\Lambda}^{\rm RS}(2.5\, r_0^{-1}) = [1.17 \pm 0.08({\rm latt.}) \pm 0.13({\rm th.}) \pm 0.09(\Lambda_{\overline{\rm MS}})]\, r_0^{-1}\,.}$$

$$\boxed{m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}}) = [4191 \pm 29({\rm latt.}) \pm 47({\rm th.}) \pm 1(\Lambda_{\overline{\rm MS}})]\,{\rm MeV}\,.}$$

## Perturbative running of $1/a$

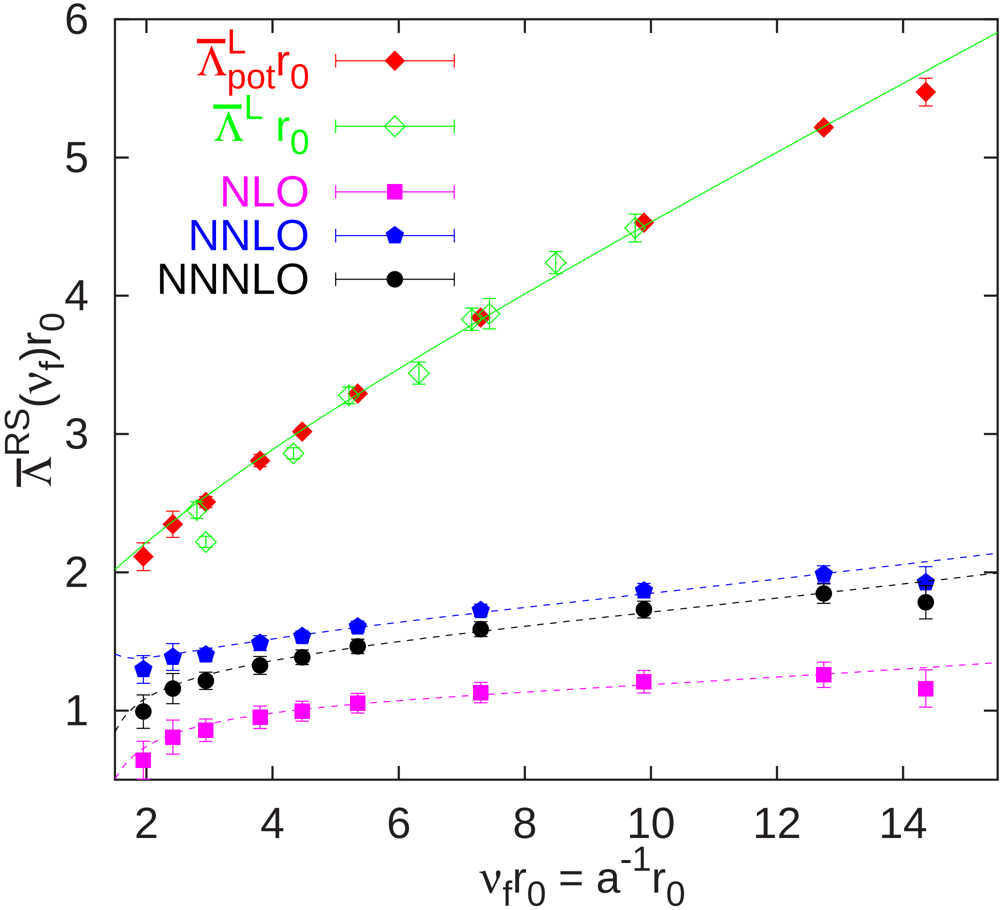
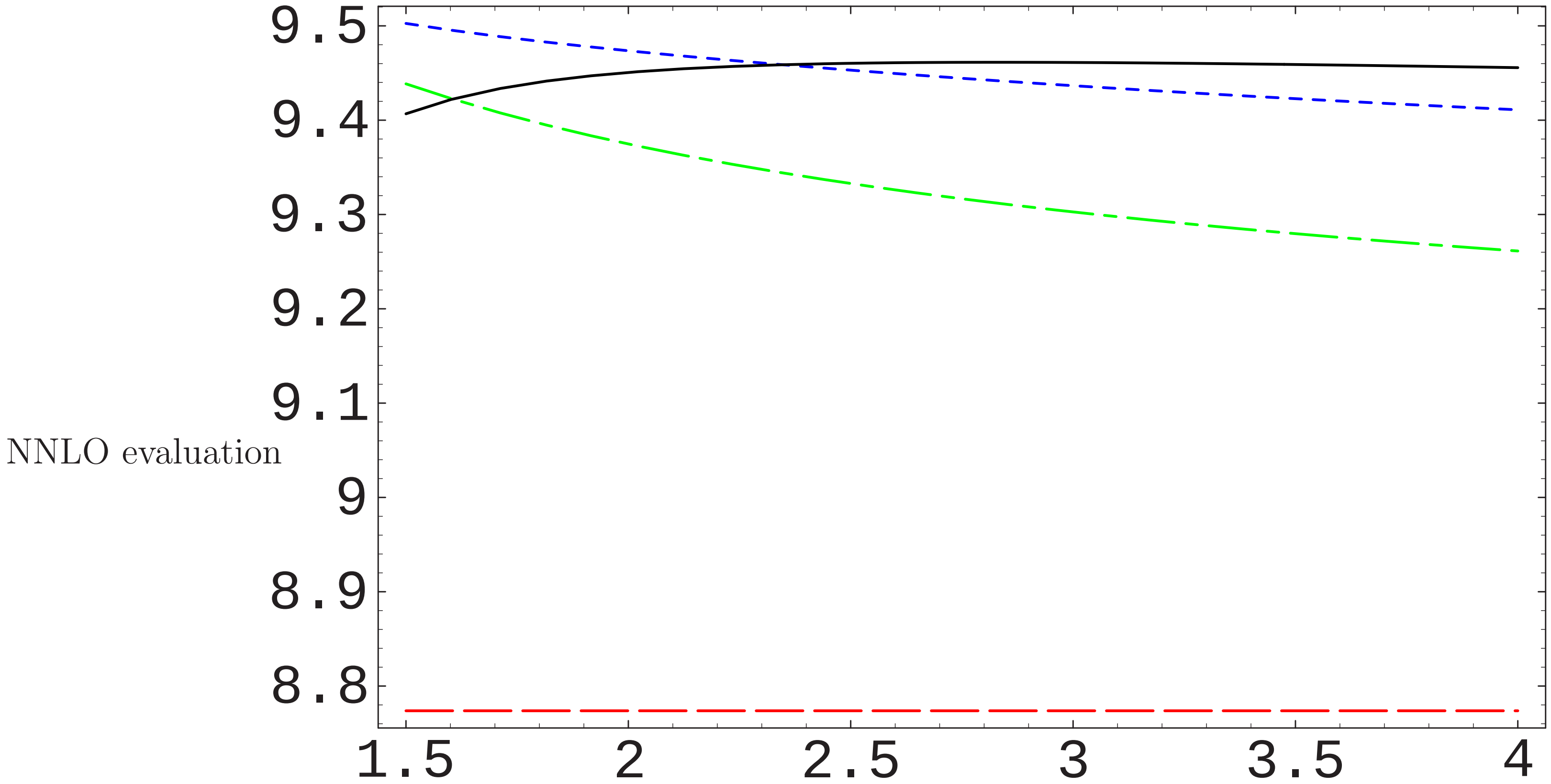



Figure 16: The binding energy $\overline{\Lambda}^{L}_{\rm pot}$, in the lattice scheme (full diamonds), in comparison with $\overline{\Lambda}^{L}$ (open diamonds). The constant $\Delta$ has been adjusted by requiring agreement between the two data sets at $r_0 \approx 7.3\,a$. The uncertainty of $\Delta = (0.988 \pm 0.067)\,r_0^{-1}$ is not included into the errors. NLO, NNLO and NNNLO refer to transformations of $\overline{\Lambda}^{L}_{\rm pot}$ into the RS scheme to different orders in perturbation theory. The solid line corresponds to the NNNLO expectation with $\Lambda_{\overline{\rm MS}} \approx 0.602\,r_0^{-1}$, and the central value, $\overline{\Lambda}^{\rm RS}(\nu_f = 9.76\,r_0^{-1}) = 1.70\,r_0^{-1}$. Bali-Pineda 2004.

## Bottom $\overline{\rm MS}$ quark mass determination from $M(\Upsilon(1S))$




$$m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}}) = 4\,210^{+90}_{-90}(\text{theory})^{-25}_{+25}(\alpha_s)\ \text{MeV}\,.$$

## Charm $\overline{\rm MS}$ quark mass determination

Weakly sensitive to long distance physics observable.

$$\langle M_B\rangle - \langle M_D\rangle = m_{b,\rm RS} - m_{c,\rm RS} + \lambda_1\left(\frac{1}{2m_{b,\rm RS}} - \frac{1}{2m_{c,\rm RS}}\right) + O(1/m_{\rm RS}^2)\,,$$

RS scheme ($\alpha_s(M_z) = 0.118 \pm 0.003$, $N_m = 0.552 \pm 0.0552$, $\lambda_1 = 0.3 \pm 0.2$ and $m_{b,\overline{\rm MS}} = 4\,203^{+67}_{-67}$ MeV)

$$m_{c,\rm RS}(1\ \text{GeV}) = 1\,181^{+82}_{-84}(m_{b,\overline{\rm MS}})^{+4}_{-1}(\alpha_s)^{-50}_{+50}(N_m)^{-78}_{+65}(\lambda_1)\ \text{MeV};$$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,206^{+66}_{-67}(m_{b,\overline{\rm MS}})^{+1}_{-0}(\alpha_s)^{+11}_{-13}(N_m)^{-62}_{+52}(\lambda_1)\ \text{MeV},$$

RS' scheme ($m_{b,\overline{\rm MS}} = 4\,214^{+67}_{-67}$ MeV)

$$m_{c,\rm RS'}(1\ \text{GeV}) = 1\,477^{+79}_{-80}(m_{b,\overline{\rm MS}})^{+34}_{-27}(\alpha_s)^{-19}_{+19}(N_m)^{-54}_{+48}(\lambda_1)\ \text{MeV};$$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,207^{+65}_{-64}(m_{b,\overline{\rm MS}})^{-7}_{+5}(\alpha_s)^{+13}_{-14}(N_m)^{-43}_{+39}(\lambda_1)\ \text{MeV}.$$

The perturbative relation between the RS and $\overline{\rm MS}$ charm mass is convergent.

$$\begin{aligned} m_{c,\rm RS}(1\ \text{GeV}) &= 1\,206 - 53 + 20 + 6 + 3 = 1\,181\ \text{MeV}, \\ m_{c,\rm RS'}(1\ \text{GeV}) &= 1\,207 + 205 + 46 + 13 + 6 = 1\,477\ \text{MeV}\,. \end{aligned}$$

Other sources of error. (1) $\pm 40(20)$ MeV error to the RS'(RS) evaluation due to the conversion from the $\overline{\rm MS}$ to the RS'(RS) bottom quark mass. (2) $1/m^2$ terms. $\sim 15(30)$ MeV error to the RS'(RS) evaluation.

Our final result reads

$$\boxed{m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,210^{+70}_{-70}(\text{theory})^{+65}_{-65}(m_{b,\overline{\rm MS}})^{-45}_{+45}(\lambda_1)\ \text{MeV}.}$$

# Conclusions

- Complete factorization of scales within dimensional regularization for nonrelativistic systems (by using effective field theories: HQET, pNRQCD, ...).
- Renormalon free working scheme preserving the power counting rules.
  - Very good description of the (lattice) static singlet and octet potential at short distances with perturbation theory.
  - Running of $1/a$ can be predicted by perturbation theory.
  - Determination of the gluelump masses, $\bar{\Lambda}$ and $m_b$ mass from lattice. Very good agreement with the direct determinations from phenomenology.

$$\Lambda_B^{\rm RS}(2.5\, r_0^{-1}) = [2.25 \pm 0.10({\rm latt.}) \pm 0.21({\rm th.}) \pm 0.08(\Lambda_{\overline{\rm MS}})]\, r_0^{-1}\,,$$

$$\Lambda_B^{\rm RS}(1\,{\rm GeV}) = [887 \pm 39({\rm latt.}) \pm 83({\rm th.}) \pm 32(\Lambda_{\overline{\rm MS}})]\,{\rm MeV}\,.$$

$$\overline{\Lambda}^{\rm RS}(2.5\, r_0^{-1}) = [1.17 \pm 0.08({\rm latt.}) \pm 0.13({\rm th.}) \pm 0.09(\Lambda_{\overline{\rm MS}})]\, r_0^{-1}\,,$$

$$m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}}) = [4191 \pm 29({\rm latt.}) \pm 47({\rm th.}) \pm 1(\Lambda_{\overline{\rm MS}})]\,{\rm MeV}\,.$$

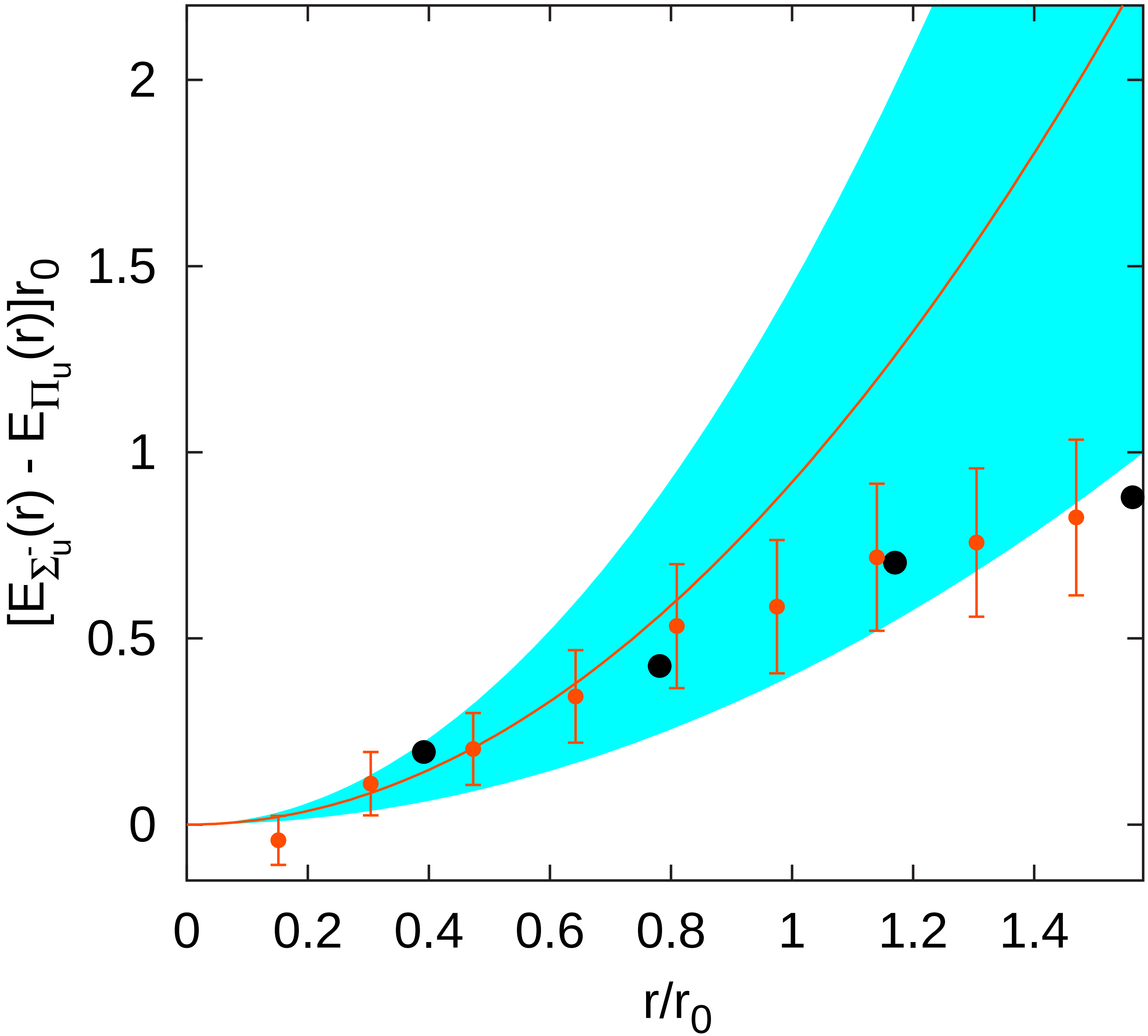



Figure 17: Splitting between the $\Sigma_u^-$ and the $\Pi_u$ potentials, extrapolated to the continuum limit, and the comparison with a quadratic fit to the $r \lesssim 0.5\,r_0$ data points ($r_0^{-1} \approx 0.4$ GeV). The big circles correspond to the data of Juge et al., obtained at finite lattice spacing $a_\sigma \approx 0.39\,r_0$. The errors in this case are smaller than the symbols.

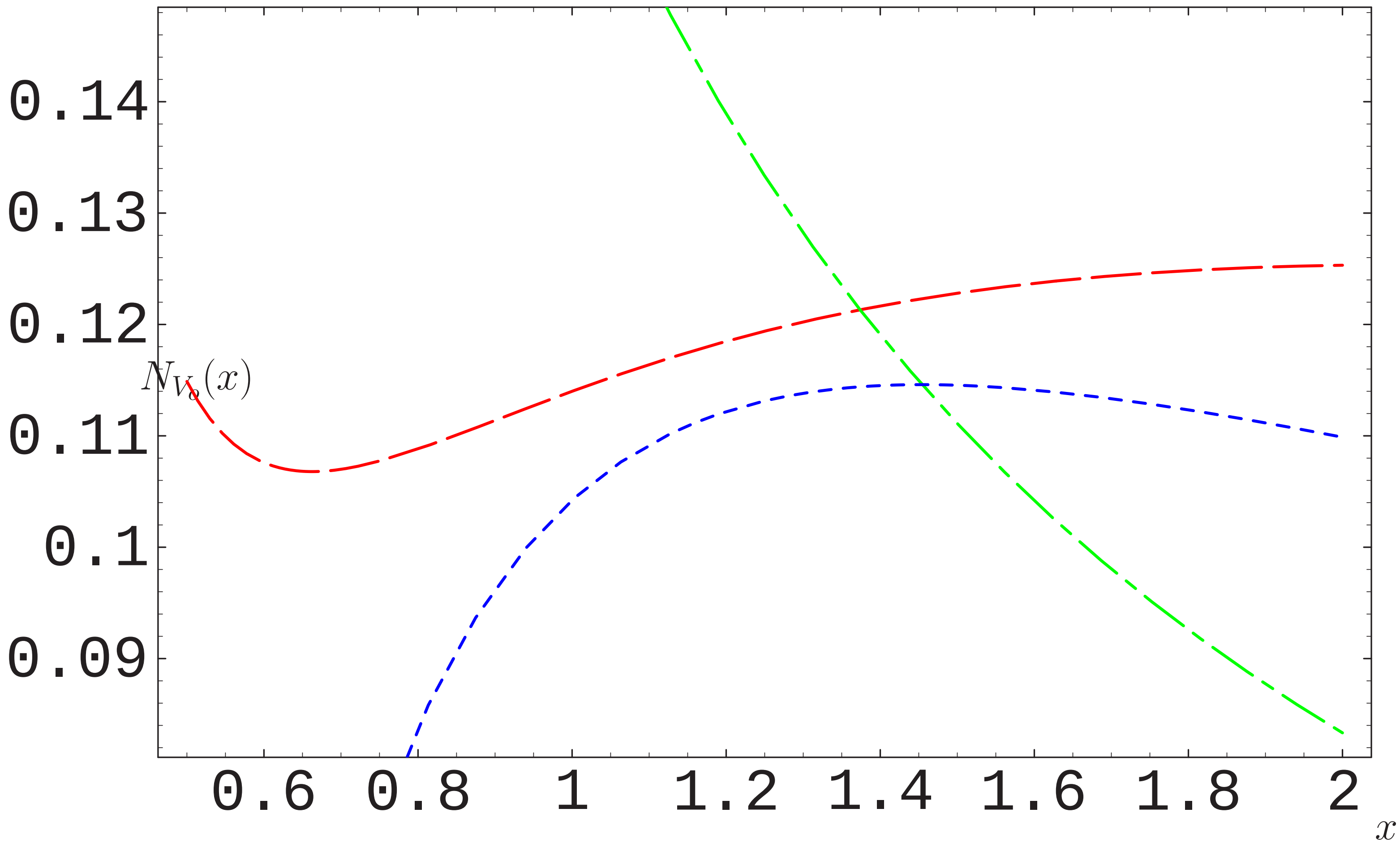
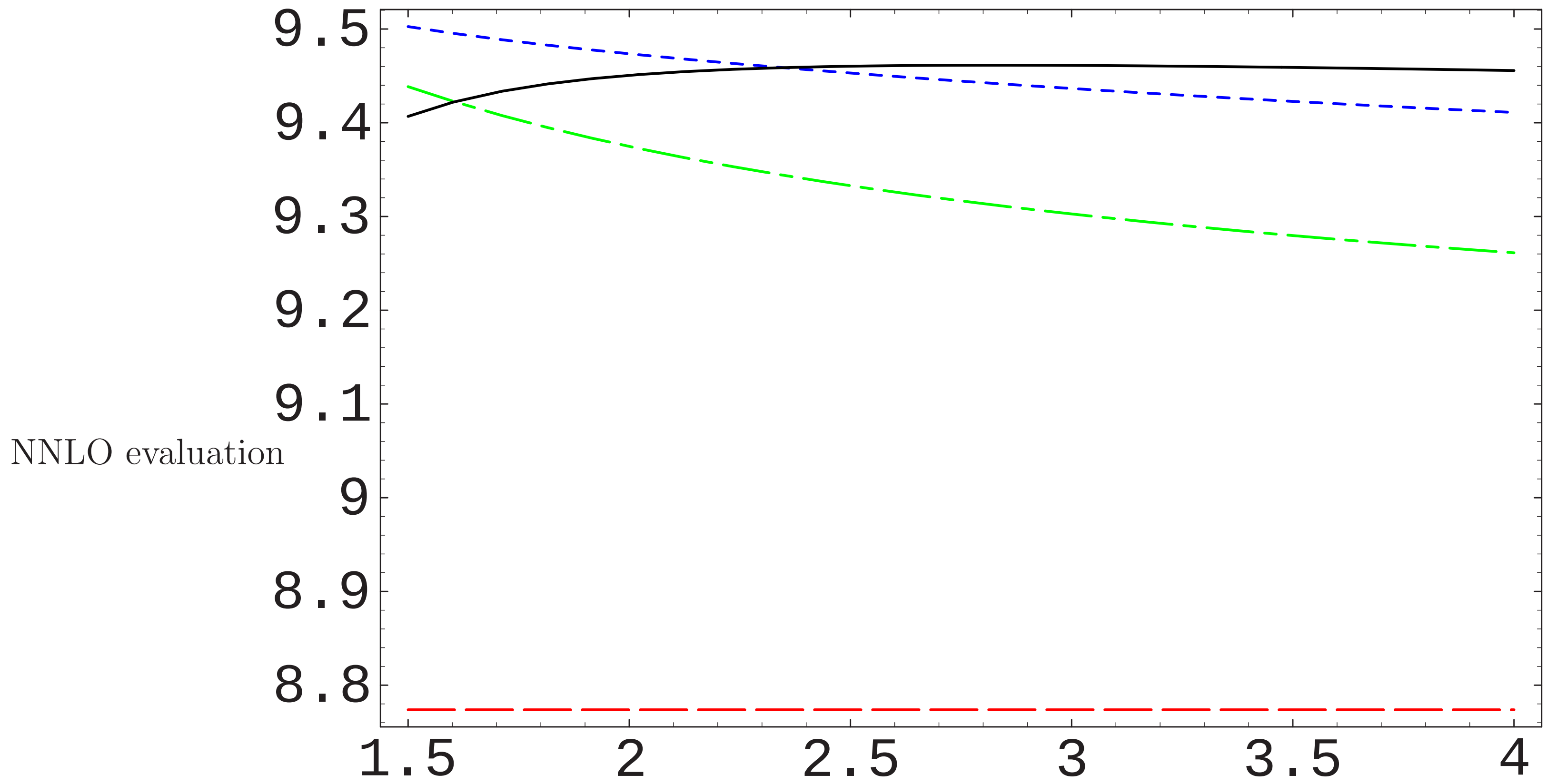



Figure 18: $x \equiv \nu r$ *dependence of* $N_{V_o}$ *for* $n_f = 0$ *at LO (dashed-dotted line), NLO (dotted line) and NNLO (dashed line).*

# Bottom $\overline{\text{MS}}$ quark mass determination from $M(\Upsilon(1S))$




Dependence on the parameters for the RS scheme: $\nu = 2.5^{+1.5}_{-1}$ GeV, $\nu_f = 2 \pm 1$ GeV, $\alpha_s(M_z) = 0.118 \pm 0.003$ and $N_m = 0.552 \pm 0.0552$

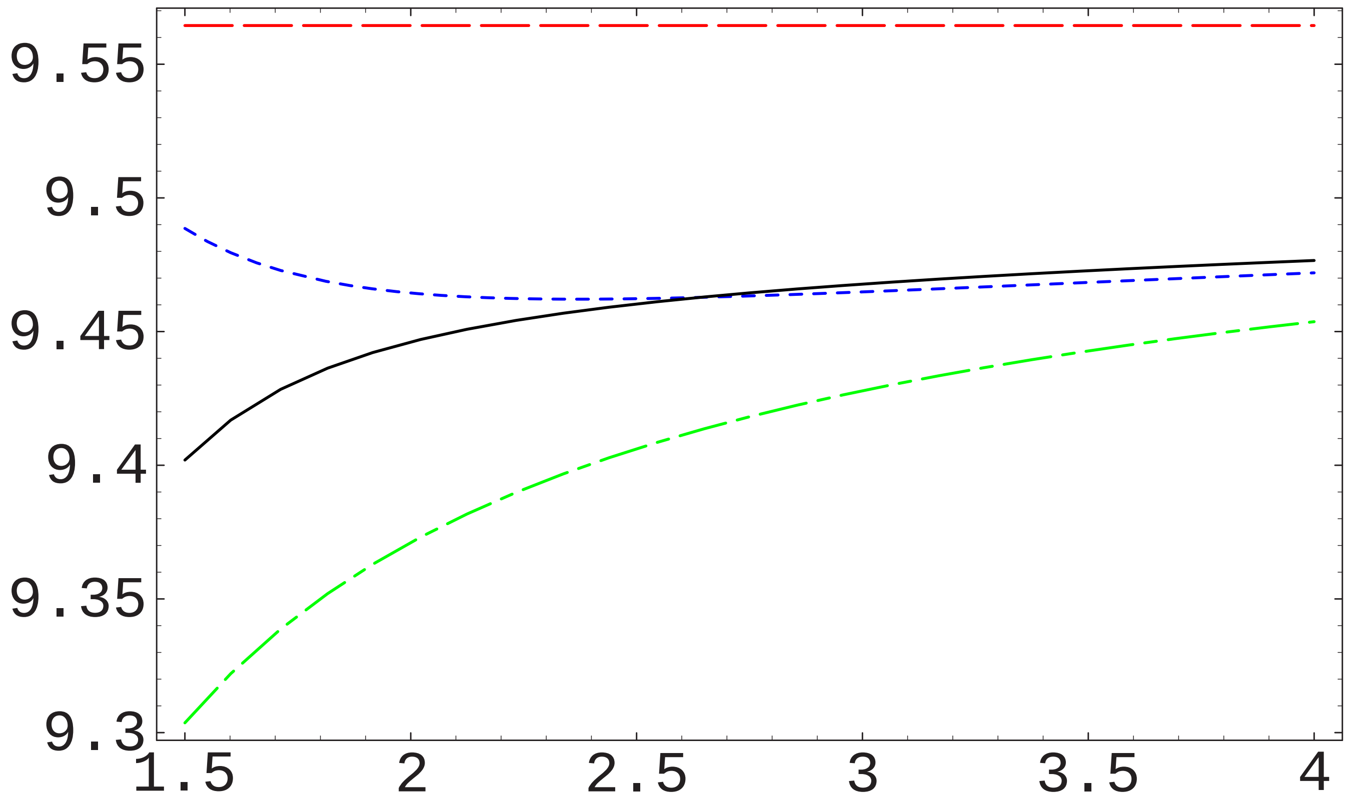
$$m_{b,\text{RS}}(2\text{ GeV}) = 4\,387^{+2}_{+28}(\nu)^{-5}_{+7}(\nu_f)^{-16}_{+16}(\alpha_s)^{-68}_{+68}(N_m)\text{ MeV};$$

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = 4\,203^{+2}_{+25}(\nu)^{-5}_{+6}(\nu_f)^{-28}_{+27}(\alpha_s)^{-10}_{+10}(N_m)\text{ MeV}.$$

Convergence. In the RS scheme

$$M_{\Upsilon(1S)} = 8\,774 + 559 + 120 + 7\text{ MeV}.$$

NNLO(st. pot.) $\sim +62$ MeV. NNLO(rel.) $\sim -55$ MeV.




For the RS' scheme, we obtain the result

$$m_{b,\mathrm{RS}'}(2\ \mathrm{GeV}) = 4\,782^{-08}_{+31}(\nu)^{-7}_{+3}(\nu_f)^{+15}_{-12}(\alpha_s)^{-28}_{+28}(N_m)\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,214^{-08}_{+28}(\nu)^{-6}_{+3}(\nu_f)^{-25}_{+25}(\alpha_s)^{-9}_{+9}(N_m)\ \mathrm{MeV}.$$

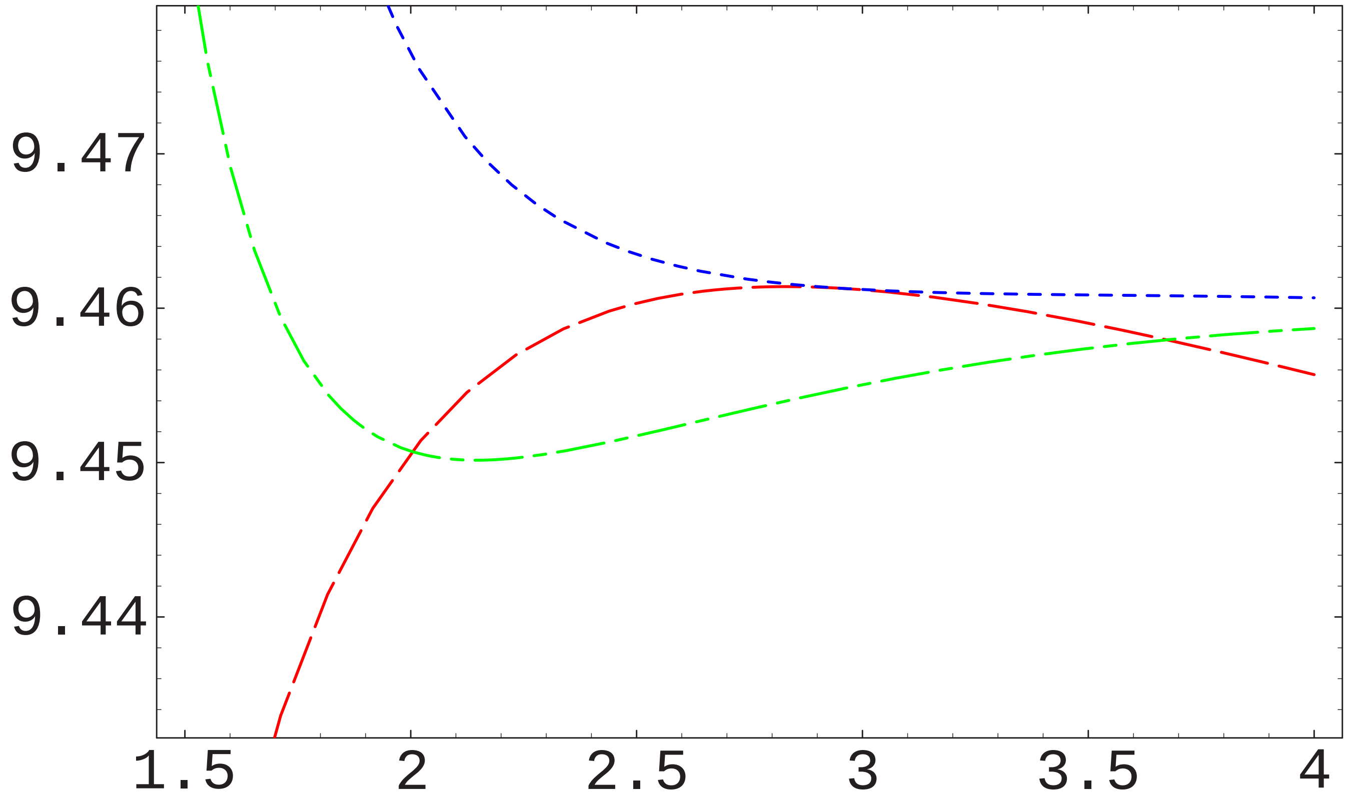
Convergence. In the RS' scheme

$$M_{\Upsilon(1S)} = 9\,564 - 158 + 56 - 2\ \mathrm{MeV}\,.$$

NNLO(st. pot.) $\sim +45$ MeV. NNLO(rel.) $\sim -47$ MeV.

NNNLO evaluation. Exact scale dependence + large $\beta_0$ estimate for the log-independent piece of $A_{101}^{5,\mathrm{OS}}$.

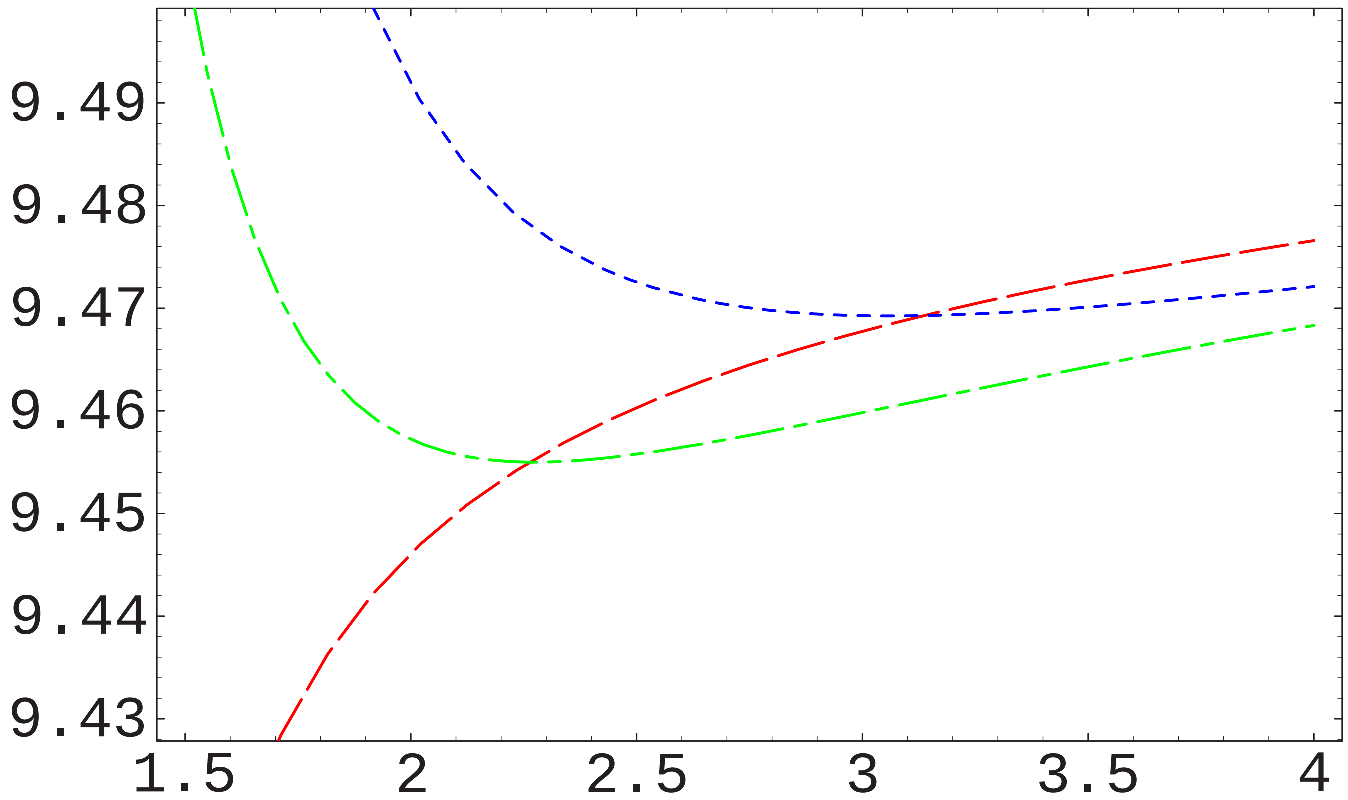
Two kind of logs: (a) $\sim \ln(m\alpha_s/\nu)$ and (b) hard/ultrasoft logs $\sim \ln(m)$, $\ln(\nu_{us})$.




Strong scale dependence at small $\nu$ (renormalization group improvement needed?)

NNNLO contribution seems to be under control.

Error $\sim \pm 50$ MeV (including charm mass effects).


9.49
9.48
9.47
9.46
9.45
9.44
9.43
1.5
2
2.5
3
3.5
4

What about US effects?

$$\delta M_{nlj}^{\rm US}(\nu_{us}) \simeq \frac{T_F}{3N_c}\int_0^\infty dt \langle n,l|\mathbf{r}e^{-t(H_o^{\rm RS}-E_n^{\rm RS})}\mathbf{r}|n,l\rangle$$
$$\times\langle g\mathbf{E}^a(t)\phi(t,0)^{\rm adj}_{ab} g\mathbf{E}^b(0)\rangle(\nu_{us}),$$

where $H_o^{\rm RS}\equiv\frac{\mathbf{p}^2}{m_{\rm RS}}+\frac{1}{2N_c}\frac{\alpha_s}{r}$ and $E_n^{\rm RS}\equiv -m_{\rm RS}C_f^2\alpha_s^2/(4n^2)$.

Situation $\Lambda_{QCD}\ll m\alpha_s^2$

$$\delta M_{nl}^{\rm US}(\nu_{us}) = \delta M_{nl}^{\rm US,\,pert.} + \delta M_{nl}^{\rm US,\,no-pert.}\,,$$

$$\delta M_{nl}^{\rm US,\,pert.} \sim m_{\rm RS}\alpha_s^5 \ln\frac{\nu_{us}}{m_{\rm RS}\alpha_s}\,,$$

$$\delta M_{nl}^{\rm US,\,no-pert.} = \sum_{n=0}^{\infty} C_n O_n\,,$$

where $C_n\sim 1/(m_{\rm RS}^{3+2n}\alpha_s^{4+4n})$ and $O_n\sim\Lambda_{QCD}^{4+2n}$.

Problems: (1) Dependence on $\alpha_s(m\alpha_s^2)$. $\langle H_o - E_1\rangle_{10}\sim 360$ MeV (up to numerical factors). (2) Convergence of the OPE (?).

$$C_0O_0 + C_1O_1 = 144 - 143\ \mathrm{MeV}\,.$$

The situation improves by lowering the scale $\nu$ ($97-66$ MeV for $\nu=2$ GeV and $53-21$ MeV for $\nu=1.5$ GeV) and it also depends on the poorly known values of the condensates.

Situation $\Lambda_{QCD}\sim m\alpha_s^2$. Non-local condensate. Basically unknown. Formally NNLO effect ($\sim 50$ MeV)

Error $\sim\pm 100$ MeV.

## Final results (RS= $N_m + \nu_f$)

RS scheme

$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = 4\,387^{+75}_{-75}(\mathrm{US/N^3LO})^{-16}_{+16}(\alpha_s)^{-73}_{+75}(\mathrm{RS})\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,203^{+67}_{-67}(\mathrm{US/N^3LO})^{-28}_{+27}(\alpha_s)^{-15}_{+16}(\mathrm{RS})\ \mathrm{MeV}.$$

RS' scheme

$$m_{b,\mathrm{RS'}}(2\ \mathrm{GeV}) = 4\,782^{+75}_{-75}(\mathrm{US/N^3LO})^{+15}_{-12}(\alpha_s)^{-35}_{+31}(\mathrm{RS'})\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,214^{+67}_{-67}(\mathrm{US/N^3LO})^{-25}_{+25}(\alpha_s)^{-15}_{+12}(\mathrm{RS'})\ \mathrm{MeV}.$$

We average the two values obtained for the $\overline{\mathrm{MS}}$ mass. We then obtain (rounding)

$$\boxed{m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,210^{+90}_{-90}(\mathrm{theory})^{-25}_{+25}(\alpha_s)\ \mathrm{MeV}\,,}$$

Check. Calculation in the OS scheme + renormalon cancellation ($\nu = m_{c,\overline{\rm MS}}$). $m_{b,\overline{\rm MS}} = 4\,210 \pm 90$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,254^{+85}_{-84}(m_{b,\overline{\rm MS}})^{+17}_{-12}(\alpha_s)^{-49}_{+45}(\lambda_1)\ {\rm MeV}\,.$$

$$m_{b,{\rm OS}} - m_{c,{\rm OS}} = 2\,956 + 490 - 14 - 32 + 22 = 3\,423\ {\rm MeV}\,.$$

Potential large logs. With $\nu = 2m_{c,\overline{\rm MS}}$, we obtain

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,239{\rm MeV}$$

and the expansion seems to improve:

$$m_{b,{\rm OS}} - m_{c,{\rm OS}} = 2\,971 + 345 + 79 + 19 + 10 = 3\,424\ {\rm MeV}.$$

## Determination of $\bar{\Lambda}$

$$\bar{\Lambda}_{\rm RS} = \langle M_B \rangle - m_{b,\rm RS} - \frac{\lambda_1}{2m_{b,\rm RS}} + O(1/m_{b,\rm RS}^2)\,.$$

We obtain (using $m_{b,\overline{\rm MS}} = 4\,210$ MeV)

$$\bar{\Lambda}_{\rm RS}(1\ {\rm GeV}) = 659\ {\rm MeV}, \qquad \bar{\Lambda}_{\rm RS'}(1\ {\rm GeV}) = 401\ {\rm MeV}.$$

We can see that it is crucial to specify the scheme in order to give a meaningful prediction for $\bar{\Lambda}$.