NONRELATIVISTIC EFFECTIVE FIELD THEORIES

AND

RENORMALONS

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Renormalons

Renormalons

They are a potential problem in effective field theories of QCD (OPE) where the matching coefficients can be computed in perturbation theory.

Examples:

OPE

NRQCD

HQET

pNRQCD

SCET

Renormalons appear as soon as we have factorization between different scales: They can worsen the convergence of the perturbative series in QCD.

Can one understand the renormalon within an effective field theory/factorization formalism?

Problems:

1) Fix the parameters of the Standard Model. Search for weakly sensitive to long distance physics observables. One wants to avoid spurious dependence on the renormalon.

2) Meaningful determination of non-perturbative parameters.

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n$$

Matching coefficients suffer from renormalon ambiguities that cancel with the ones of the matrix elements in effective field theory calculations.

$$c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

Its Borel transform would be

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^{\infty} \mathrm{d}t \, e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the matching coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

$$\delta B[c](t) \sim \frac{1}{a-t},$$

where a is a positive number, it sets up the maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation, which is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*},$$

where $n^* \sim \frac{a}{\alpha_s}$. Moreover, the fact that a is positive means that, even after Borel resummation, c suffers from a non-perturbative ambiguity of order

$$\delta c \sim \left(\Lambda_{QCD}\right)^{\frac{a\beta_0}{2\pi}}.$$

Examples (only true if the perturbative piece can be computed with such precision)

$$\delta_{np} M_B \sim \Lambda_{QCD}, \; \delta_{np}\Gamma(B \to X_u l\nu) \sim G_F^2 m_{\rm OS}^3 \Lambda_{QCD}^2,$$

$$\delta_{np} M_{\Upsilon(1S)} \sim m_{\rm OS} \frac{\Lambda_{QCD}^4}{(m_{\rm OS}\alpha_s)^4},$$

Problem. The OS mass suffers from renormalon ambiguities:

$$\delta_{np}^{(\rm pert.)} m_{\rm OS} = \delta_{np}^{(\rm pert.)} m_{\overline{\rm MS}} (1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{QCD}!$$

and the precision of perturbation theory is

$$\delta_{np}^{(\rm pert.)} M_B \sim \Lambda_{QCD}, \; \delta_{np}^{(\rm pert.)}\Gamma \sim G_F^2 m_{\rm OS}^4 \Lambda_{QCD},$$

$$\delta_{np}^{(\rm pert.)} M_{\Upsilon(1S)} = \delta_{np}^{(\rm pert.)} m_{\rm OS}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \cdots) \sim \Lambda_{QCD}.$$

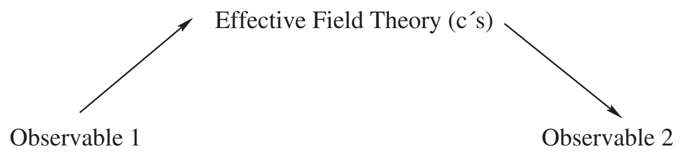


Figure 1: *Symbolic relation between observables through the determination of the matching coefficients of the effective field theory.*

Proposal: to subtract the renormalon from the matching coefficients.

Physics. Computations to n-loops produce small scales: me^{-n}. From the effective field theory point of view these scales should be in the effective field theory instead that in the matching coefficients.

Future: Factorization in dimensional regularization?

OS mass

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1} \,,$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\rm OS}](t(u)) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (\text{analytic term}),$$

Next renormalon at $u = 1$.

$$r_n \stackrel{n\to\infty}{=} N_m\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right).$$

$$b = \frac{\beta_1}{2\beta_0^2}, \qquad c_1 = \frac{1}{4\, b \beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right), \qquad \cdots$$

Determination of N_m

$$\begin{aligned} D_m(u) &= \sum_{n=0}^{\infty} D_m^{(n)} u^n = (1-2u)^{1+b} B[m_{\rm OS}](t(u)) \\ &= N_m \nu \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (1-2u)^{1+b} (\text{analytic term})\,. \end{aligned}$$

$$N_m \nu = D_m(u = 1/2).$$

$$\begin{aligned} N_m &= 0.4244 + 0.1379 + 0.0127 = 0.5750 \quad (n_f = 3) \\ &= 0.4244 + 0.1275 + 0.0004 = 0.5523 \quad (n_f = 4) \\ &= 0.4244 + 0.1199 - 0.0208 = 0.5235 \quad (n_f = 5) \end{aligned}$$

$\nu \sim m$

Large β_0 analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1 + (2u-1)\ln\frac{\nu}{m} + \cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u - 1 \ll 1$)

$$(2u-1)\ln\frac{\nu}{m} \ll 1\,.$$

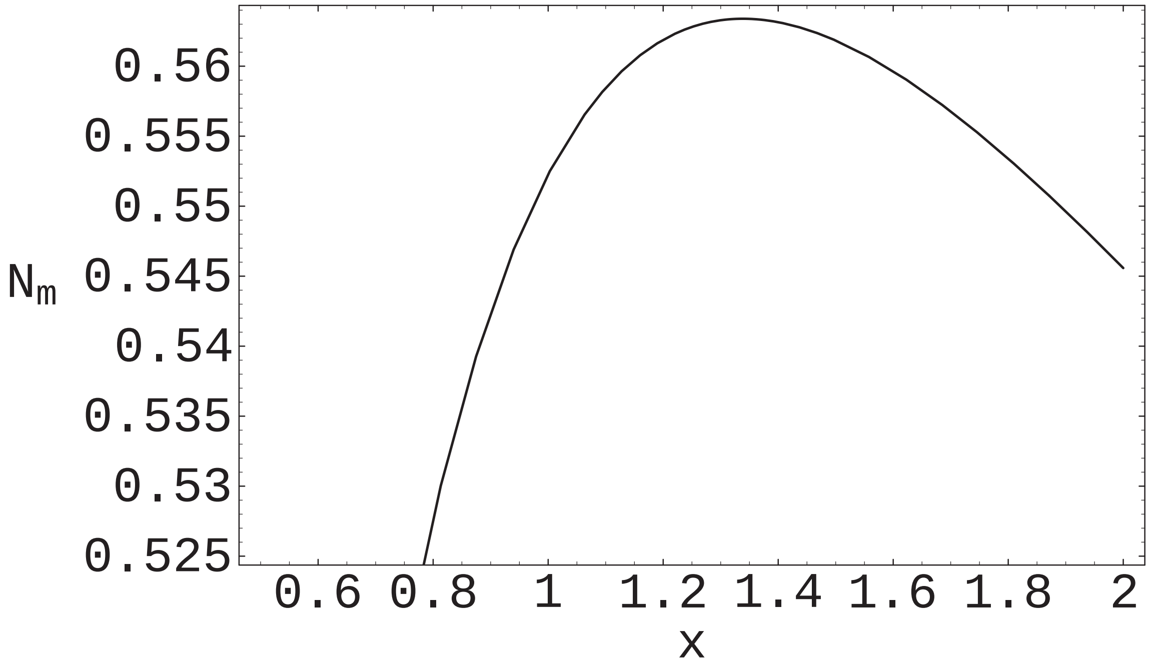


Figure 2: $x \equiv \frac{\nu}{m_{\overline{\rm MS}}}$ *dependence of* N_m *for* $n_f = 4$.

Estimates of r_n

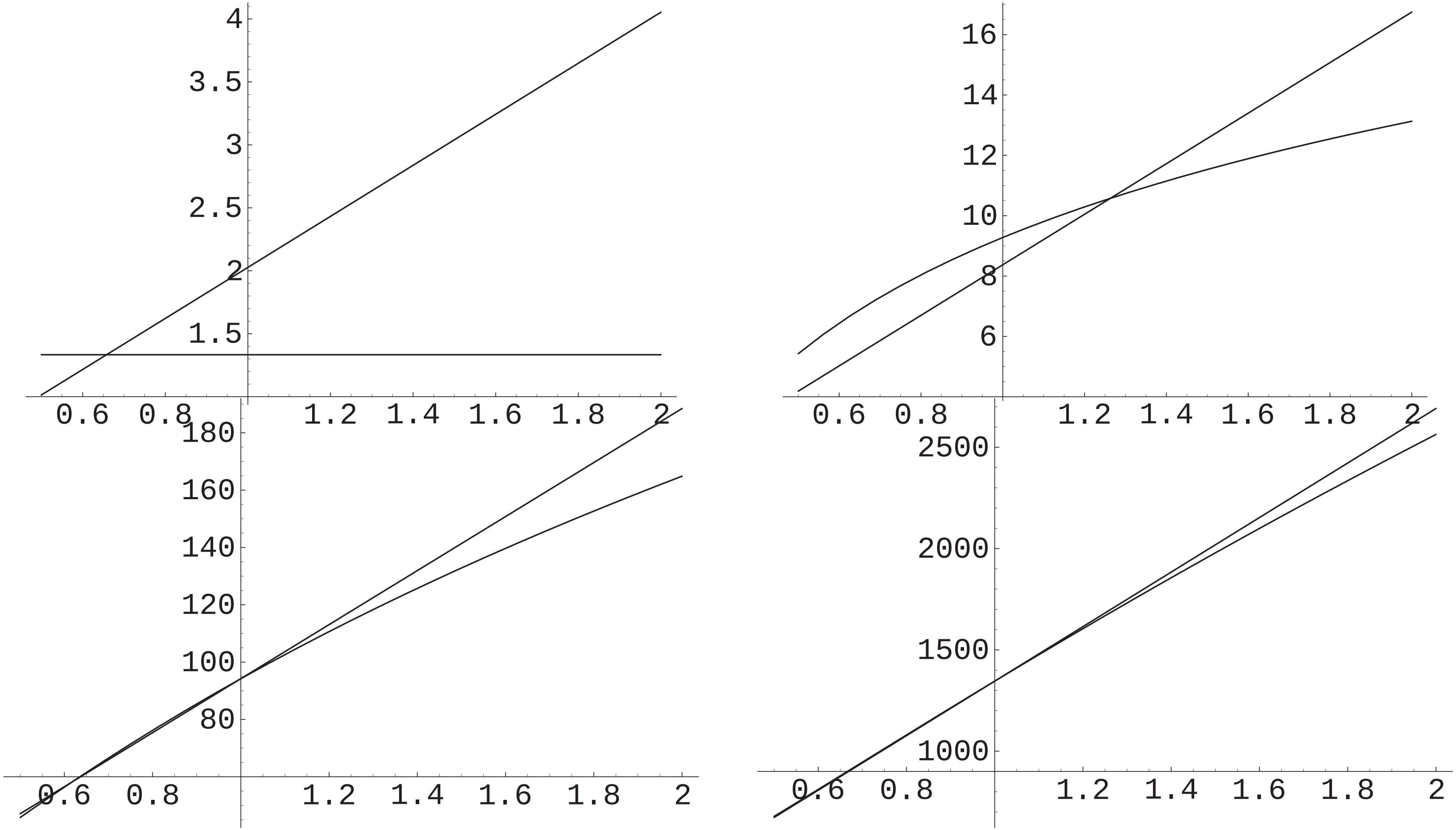
$\tilde{r}_n = r_n/m_{\overline{\rm MS}}$	$\tilde{r}_0$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$
exact ($n_f = 3$)	0.424413	1.04556	3.75086	$---$	$---$
our estimate ($n_f = 3$)	0.617148	0.977493	3.76832	18.6697	118.441
large β_0 ($n_f = 3$)	0.424413	1.42442	3.83641	17.1286	97.5872
exact ($n_f = 4$)	0.424413	0.940051	3.03854	$---$	$---$
our estimate ($n_f = 4$)	0.645181	0.848362	3.03913	13.8151	80.5776
large β_0 ($n_f = 4$)	0.424413	1.31891	3.28911	13.5972	71.7295
exact ($n_f = 5$)	0.424413	0.834538	2.36832	$---$	$---$
our estimate ($n_f = 5$)	0.706913	0.713994	2.36440	9.73117	51.5952
large β_0 ($n_f = 5$)	0.424413	1.21339	2.78390	10.5880	51.3865

Table 1: *Values of r_n for $\nu = m_{\overline{\rm MS}}$. Either the exact result (when available), our estimate, or the estimate using the large β_0 approximation.*

$\tilde{r}_n = r_n/m_{\overline{\rm MS}}$	$\tilde{r}_0$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$
$O(1/n)$ ($n_f = 3$)	-0.164	-0.046	-0.027	-0.019	-0.015
$O(1/n^2)$ ($n_f = 3$)	0.237	-0.103	-0.017	-0.007	-0.004
$O(1/n)$ ($n_f = 4$)	-0.105	-0.028	-0.016	-0.012	-0.009
$O(1/n^2)$ ($n_f = 4$)	0.274	-0.126	-0.020	-0.008	-0.004
$O(1/n)$ ($n_f = 5$)	0.024	0.006	0.003	0.002	0.002
$O(1/n^2)$ ($n_f = 5$)	0.326	-0.165	-0.023	-0.009	-0.005

Table 2: $O(1/n)$ *corrections (normalized with respect the leading solution) of our r_n estimates for different number of light fermions.*

$$c_1(n_f = 0) \simeq -0.215, \quad c_2(n_f = 0) \simeq 0.185$$


4
3.5
3
2.5
2
1.5
0.6 0.8 1.2 1.4 1.6 1.8 2
16
14
12
10
8
6
0.6 0.8 1.2 1.4 1.6 1.8 2
180
160
140
120
100
80
0.6 0.8 1.2 1.4 1.6 1.8 2
2500
2000
1500
1000
0.6 0.8 1.2 1.4 1.6 1.8 2

The static potential

$$V_s^{(0)}(r;\nu_{us}) = \sum_{n=0}^{\infty} V_{s,n}^{(0)}\alpha_s^{n+1},$$

$2m_{\rm OS}+V_s^{(0)}$ (not $2m_{\rm OS}+V_o^{(0)}$) can be understood as an observable up to $O(r^2\Lambda_{QCD}^3, \Lambda_{QCD}^2/m)$ renormalon (and/or non-perturbative) contributions. We can use our knowledge of the asymptotic behavior of $m_{\rm OS}$.

$$V_{s,n}^{(0)} \overset{n\to\infty}{=} N_V\,\nu\left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right)$$

$$2N_m + N_V = 0$$

$$\begin{aligned} D_V(u) &= \sum_{n=0}^{\infty} D_V^{(n)}u^n = (1-2u)^{1+b}B[V_s^{(0)}](t(u)) \\ &= N_V\nu\left(1+c_1(1-2u)+c_2(1-2u)^2+\cdots\right)+(1-2u)^{1+b}(\text{analytic term}). \end{aligned}$$

Next (IR) renormalon at $u=3/2$.

$$\begin{aligned} N_V &= -1.333+0.572-0.345=-1.107 \quad (n_f=3) \\ &= -1.333+0.585-0.329=-1.077 \quad (n_f=4) \\ &= -1.333+0.587-0.295=-1.042 \quad (n_f=5)\,. \end{aligned}$$

$$2\frac{2N_m+N_V}{2N_m-N_V} = \begin{cases} 0.038 & , n_f=3 \\ 0.025 & , n_f=4 \\ 0.005 & , n_f=5. \end{cases}$$

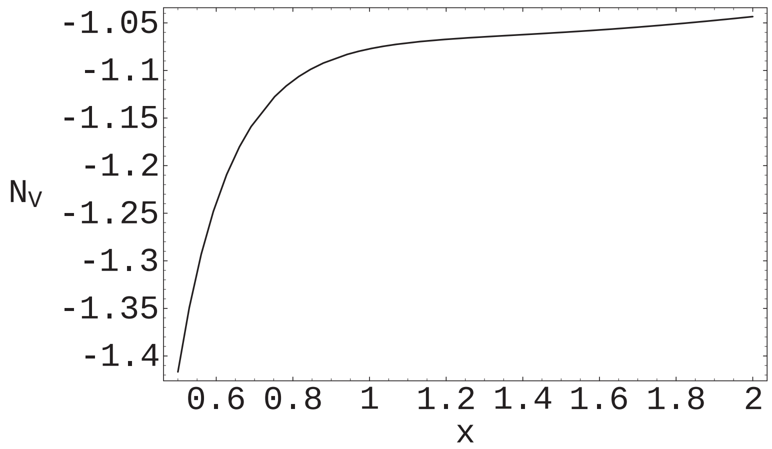


Figure 3: *$x \equiv \nu r$ dependence of N_V for $n_f = 4$.*

$\tilde{V}^{(0)}_{s,n} = rV^{(0)}_{s,n}$	$\tilde{V}^{(0)}_{s,0}$	$\tilde{V}^{(0)}_{s,1}$	$\tilde{V}^{(0)}_{s,2}$	$\tilde{V}^{(0)}_{s,3}$	$\tilde{V}^{(0)}_{s,4}$
exact ($n_f = 3$)	-1.33333	-1.84512	-7.28304	$---$	$---$
our estimate ($n_f = 3$)	-1.23430	-1.95499	-7.53665	-37.3395	-236.882
large β_0 ($n_f = 3$)	-1.33333	-2.69395	-7.69303	-34.0562	$---$
exact ($n_f = 4$)	-1.33333	-1.64557	-5.94978	$---$	$---$
our estimate ($n_f = 4$)	-1.29036	-1.69672	-6.07826	-27.6301	-161.155
large β_0 ($n_f = 4$)	-1.33333	-2.49440	-6.59553	-27.0349	$---$
exact ($n_f = 5$)	-1.33333	-1.44602	-4.70095	$---$	$---$
our estimate ($n_f = 5$)	-1.41383	-1.42799	-4.72881	-19.4623	-103.190
large β_0 ($n_f = 5$)	-1.33333	-2.29485	-5.58246	-21.0518	$---$

Table 3: *Values of $V^{(0)}_{s,n}$ with $\nu = 1/r$. Either the exact result (when available), our estimate, or the estimate using the large β_0 approximation.*

Renormalon subtracted matching and power counting

Effective field theory with renormalon free parameters but preserving the power counting rules.

The renormalon is associated to the non-analytic behavior in $1-2u$. These terms also exist in the effective theory. Procedure: to explicitly subtract them from the matching coefficients (the mass).

$$B[m_{\rm RS}] \equiv B[m_{\rm OS}] - N_m \nu_f \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right),$$

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \sum_{n=0}^{\infty} N_m \, \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}.$$

Expansion in $\alpha_s(\nu)$

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS} \alpha_s^{n+1},$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}}, \nu, \nu_f)$. They are the ones expected to be of natural size. We now do not loose accuracy if we first obtain $m_{\rm RS}$ and later on $m_{\overline{\rm MS}}$.

Different scheme

$$B[m_{RS'}] \equiv B[m_{RS}] + N_m \nu_f \left(1 + c_1 + c_2 + \cdots\right).$$

Check of convergence improvement

Masses	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
$m_{\rm OS}$	401	199	144	147	5 102
$m_{\rm RS}$	111	50	17	7	4 395
$m_{\rm RS'}$	401	114	38	15	4.778
$m_{\rm PS}$	210	80	42	$---$	4 542
$m_{1S}^{\rm (static)}$	102	50	19	8	4 389
$m_{\rm RS}$	256	95	40	21	4 622
$m_{\rm RS'}$	401	157	74	41	4.882
$m_{\rm PS}$	306	120	67	$---$	4.703
$m_{1S}^{\rm (static)}$	251	94	41	22	4 619

Table 4: *Contributions at various orders in α_s for different mass definitions for the bottom quark case, either with $\nu_f = 1/r = 2$ GeV (middle panel) or with $\nu_f = 1/r = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 1 has been used. The other parameters have been fixed to the values $m_{\overline{\rm MS}}(m_{\overline{\rm MS}}) = 4.21$ GeV, $\nu = m_{\overline{\rm MS}}(m_{\overline{\rm MS}})$ and $n_f = 4$.*

$$m_{1S}^{\rm (static)} \equiv m_{\rm OS} + \frac{V(r)}{2} = m_{\overline{\rm MS}} + \left(r_0 - \frac{C_f}{2r}\right)\alpha_s + \cdots .$$

HQET

$$\mathcal{L} = \bar{h}\left(iD_0 - \delta m_{\rm RS}\right)h + O\left(\frac{1}{m_{RS}}\right),$$

where $\delta m_{\rm RS} = m_{\rm OS} - m_{\rm RS}$ and similarly for the NRQCD Lagrangian.

Weakly sensitive to long distance physics observable

$$\langle M_B\rangle - \langle M_D\rangle = m_{b,\rm RS} - m_{c,\rm RS} + \lambda_1\left(\frac{1}{2m_{b,\rm RS}} - \frac{1}{2m_{c,\rm RS}}\right) + O(1/m_{\rm RS}^2)\,.$$

pNRQCD. If $\Lambda_{QCD} \ll m\alpha_s$

$$V^{(0)}_{s,\mathrm{RS(RS')}}(\nu_f) = V^{(0)}_s + 2\delta m_{\mathrm{RS(RS')}} \,,$$

Check of convergence improvement

Potentials	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
$V_s^{(0)}$	-910	-306	-302	-383	$-1\,902$
$V_{s,\mathrm{RS}}^{(0)}$	-205	3	-2	-3	-208
$V_{s,\mathrm{RS'}}^{(0)}$	-910	-54	-14	-6	-984
$V_{s,\mathrm{PS}}^{(0)}$	-446	-42	-25	$---$	-513
$V_{s,\mathrm{RS}}^{(0)}$	-558	-63	-41	-26	-687
$V_{s,\mathrm{RS'}}^{(0)}$	-910	-180	-95	-54	$-1\,239$
$V_{s,\mathrm{PS}}^{(0)}$	-678	-116	-75	$---$	-869

Table 5: *Contributions at various orders in α_s for different singlet static potential definitions for some typical scales in the Υ system, either with $\nu_f = 2$ GeV (middle panel) or with $\nu_f = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 3 has been used. The other parameters have been fixed to the values $\nu = 1/r = 2.5$ GeV and $n_f = 4$.*

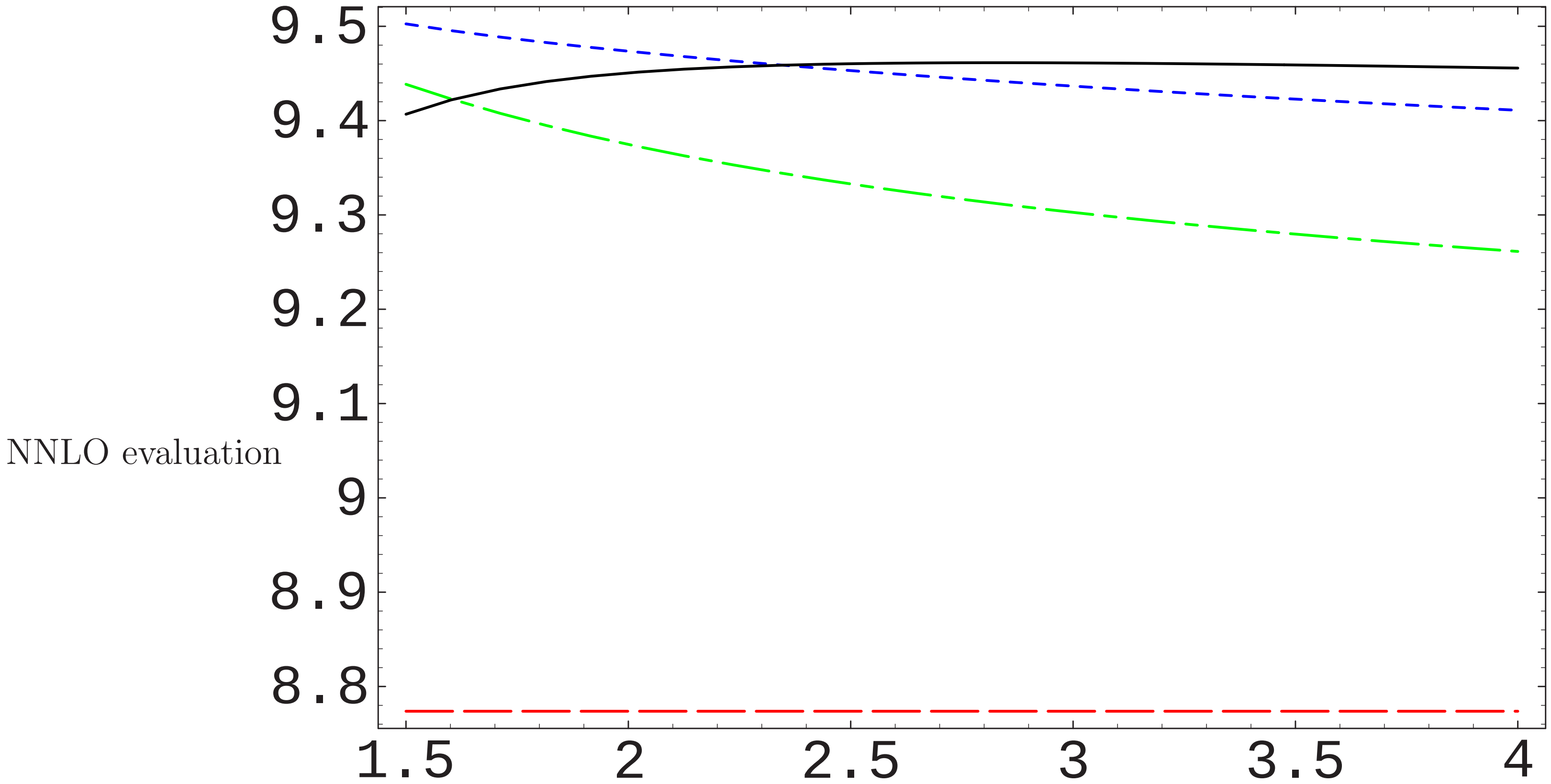
pNRQCD Lagrangian

$$\mathcal{L}^{(0)} = \mathrm{Tr}\{S^\dagger\left(i\partial_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{s,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right) S + O^\dagger\left(iD_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{o,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right) O\}$$
$$+\; gV_A(\mathbf{x})\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, S + S^\dagger \mathbf{x}\cdot\mathbf{E}\, O\right\} + g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, O + O^\dagger O \mathbf{x}\cdot\mathbf{E}\right\},$$

Weakly sensitive to long distance physics observable

$$M_{nlj} = 2m_{\mathrm{RS}} + \sum_{m=2}^{\infty} A^{m,\mathrm{RS}}_{nlj}(\nu_{us})\alpha_s^m + \delta M^{\mathrm{US}}_{nlj}(\nu_{us})\,.$$

Bottom $\overline{\mathrm{MS}}$ quark mass determination



Dependence on the parameters for the RS scheme: $\nu = 2.5^{+1.5}_{-1}$ GeV, $\nu_f = 2 \pm 1$ GeV, $\alpha_s(M_z) = 0.118 \pm 0.003$ and $N_m = 0.552 \pm 0.0552$

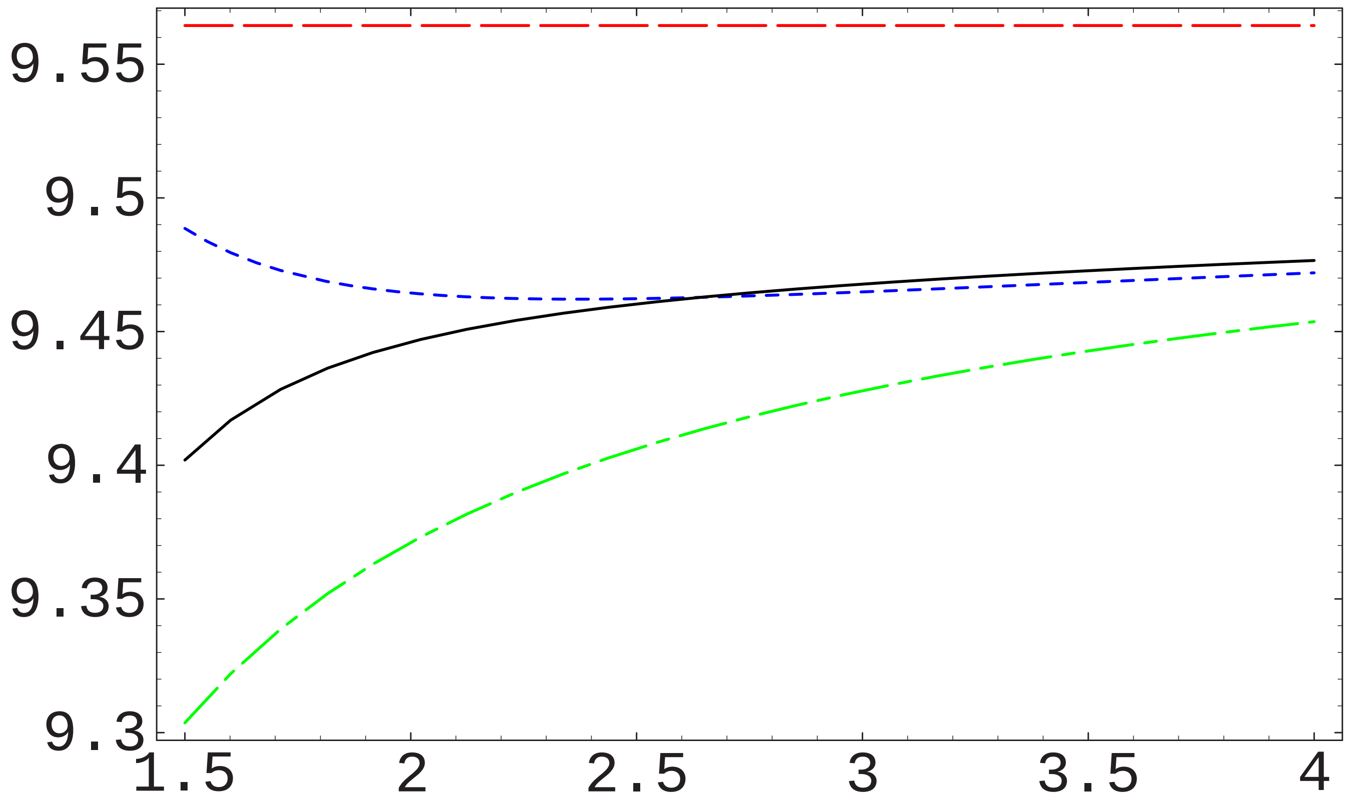
$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = 4\,387^{+2}_{+28}(\nu)^{-5}_{+7}(\nu_f)^{-16}_{+16}(\alpha_s)^{-68}_{+68}(N_m)\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,203^{+2}_{+25}(\nu)^{-5}_{+6}(\nu_f)^{-28}_{+27}(\alpha_s)^{-10}_{+10}(N_m)\ \mathrm{MeV}.$$

Convergence. In the RS scheme

$$M_{\Upsilon(1S)} = 8\,774 + 559 + 120 + 7\ \mathrm{MeV}.$$

NNLO(st. pot.) $\sim +62$ MeV. NNLO(rel.) ~ -55 MeV.



For the RS' scheme, we obtain the result

$$m_{b,\mathrm{RS}'}(2\ \mathrm{GeV}) = 4\,782^{-08}_{+31}(\nu)^{-7}_{+3}(\nu_f)^{+15}_{-12}(\alpha_s)^{-28}_{+28}(N_m)\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,214^{-08}_{+28}(\nu)^{-6}_{+3}(\nu_f)^{-25}_{+25}(\alpha_s)^{-9}_{+9}(N_m)\ \mathrm{MeV}.$$

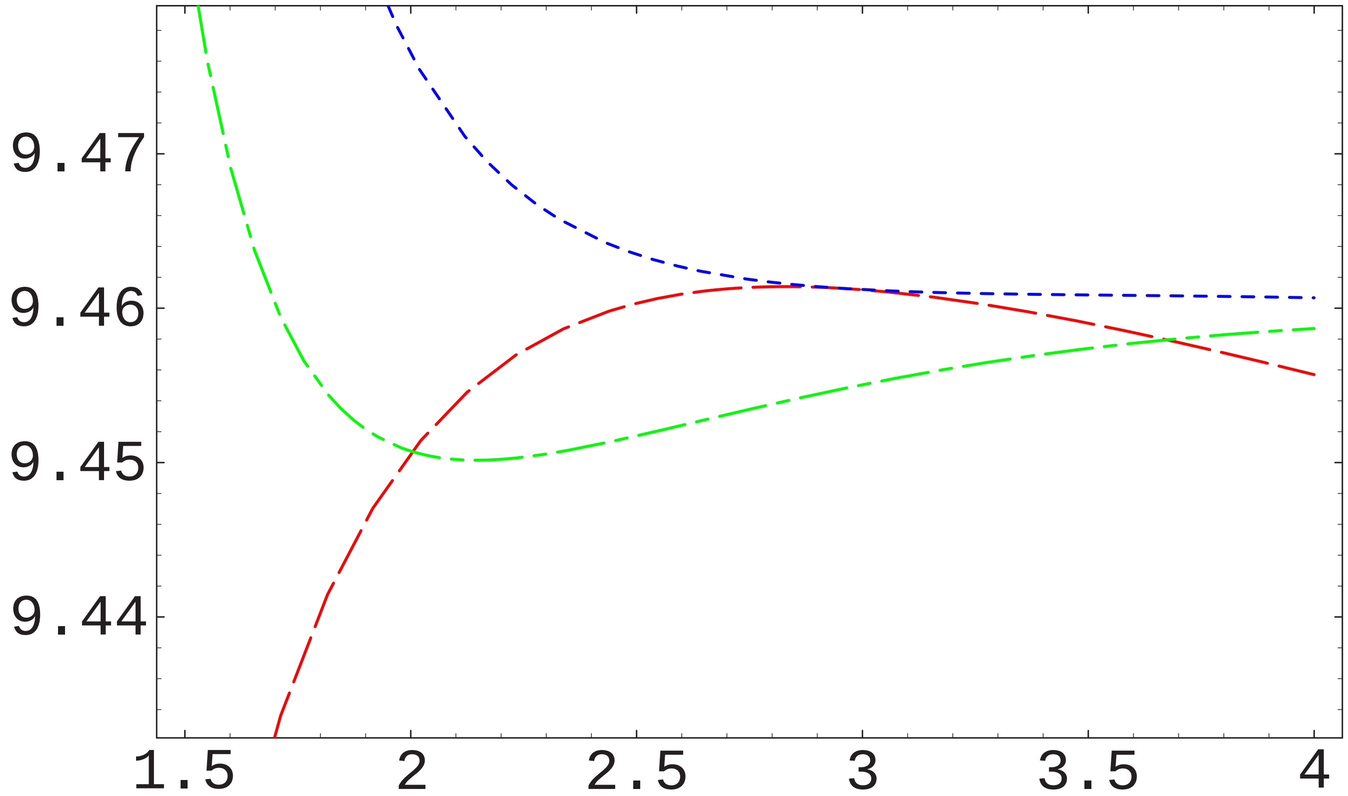
Convergence. In the RS' scheme

$$M_{\Upsilon(1S)} = 9\,564 - 158 + 56 - 2\ \mathrm{MeV}\,.$$

NNLO(st. pot.) $\sim +45$ MeV. NNLO(rel.) ~ -47 MeV.

NNNLO evaluation. Exact scale dependence + large β_0 estimate for the log-independent piece of $A_{101}^{5,\mathrm{OS}}$.

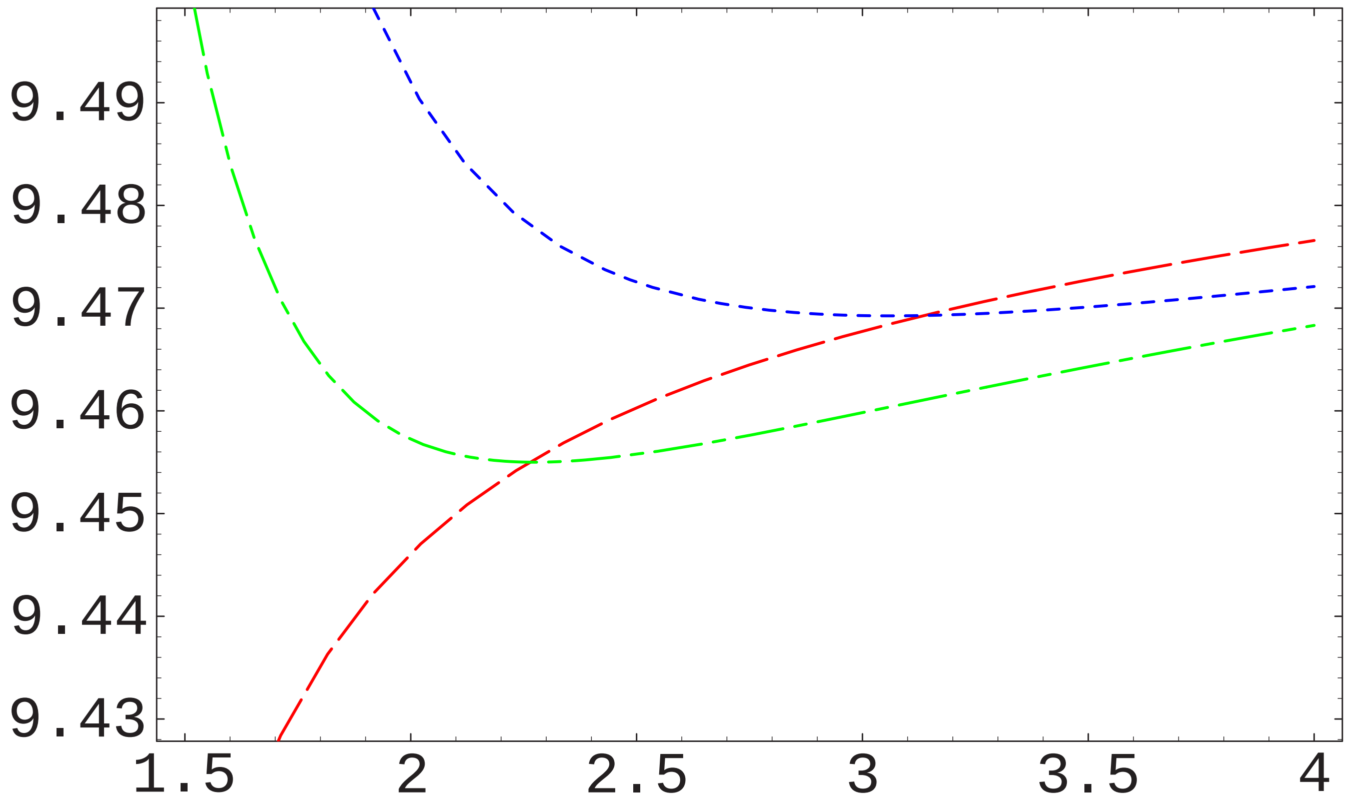
Two kind of logs: (a) $\sim \ln(m\alpha_s/\nu)$ and (b) hard/ultrasoft logs $\sim \ln(m)$, $\ln(\nu_{us})$.



Strong scale dependence at small ν (renormalization group improvement needed?)

NNNLO contribution seems to be under control.

Error $\sim \pm 50$ MeV (including charm mass effects).


9.49
9.48
9.47
9.46
9.45
9.44
9.43
1.5
2
2.5
3
3.5
4

What about US effects?

$$\delta M_{nlj}^{\rm US}(\nu_{us}) \simeq \frac{T_F}{3N_c}\int_0^\infty dt \langle n,l|\mathbf{r} e^{-t(H_o^{\rm RS}-E_n^{\rm RS})}\mathbf{r}|n,l\rangle$$
$$\times\langle g\mathbf{E}^a(t)\phi(t,0)_{ab}^{\rm adj} g\mathbf{E}^b(0)\rangle(\nu_{us}),$$

where $H_o^{\rm RS} \equiv \frac{\mathbf{p}^2}{m_{\rm RS}} + \frac{1}{2N_c}\frac{\alpha_s}{r}$ and $E_n^{\rm RS} \equiv -m_{\rm RS}C_f^2\alpha_s^2/(4n^2)$.

Situation $\Lambda_{QCD} \ll m\alpha_s^2$

$$\delta M_{nl}^{\rm US}(\nu_{us}) = \delta M_{nl}^{\rm US,\,pert.} + \delta M_{nl}^{\rm US,\,no-pert.}\,,$$

$$\delta M_{nl}^{\rm US,\,pert.} \sim m_{\rm RS}\alpha_s^5 \ln\frac{\nu_{us}}{m_{\rm RS}\alpha_s}\,,$$

$$\delta M_{nl}^{\rm US,\,no-pert.} = \sum_{n=0}^{\infty} C_n O_n\,,$$

where $C_n \sim 1/(m_{\rm RS}^{3+2n}\alpha_s^{4+4n})$ and $O_n \sim \Lambda_{QCD}^{4+2n}$.

Problems: (1) Dependence on $\alpha_s(m\alpha_s^2)$. $\langle H_o - E_1\rangle_{10} \sim 360$ MeV (up to numerical factors). (2) Convergence of the OPE (?).

$$C_0O_0 + C_1O_1 = 144 - 143\ \mathrm{MeV}\,.$$

The situation improves by lowering the scale ν ($97-66$ MeV for $\nu = 2$ GeV and $53-21$ MeV for $\nu = 1.5$ GeV) and it also depends on the poorly known values of the condensates.

Situation $\Lambda_{QCD} \sim m\alpha_s^2$. Non-local condensate. Basically unknown. Formally NNLO effect (~ 50 MeV)

Error $\sim \pm 100$ MeV.

Final results (RS$= N_m + \nu_f$)

RS scheme

$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = 4\,387^{+75}_{-75}(\mathrm{US/N^3LO})^{-16}_{+16}(\alpha_s)^{-73}_{+75}(\mathrm{RS})\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,203^{+67}_{-67}(\mathrm{US/N^3LO})^{-28}_{+27}(\alpha_s)^{-15}_{+16}(\mathrm{RS})\ \mathrm{MeV}.$$

RS' scheme

$$m_{b,\mathrm{RS'}}(2\ \mathrm{GeV}) = 4\,782^{+75}_{-75}(\mathrm{US/N^3LO})^{+15}_{-12}(\alpha_s)^{-35}_{+31}(\mathrm{RS'})\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,214^{+67}_{-67}(\mathrm{US/N^3LO})^{-25}_{+25}(\alpha_s)^{-15}_{+12}(\mathrm{RS'})\ \mathrm{MeV}.$$

We average the two values obtained for the $\overline{\mathrm{MS}}$ mass. We then obtain (rounding)

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,210^{+90}_{-90}(\mathrm{theory})^{-25}_{+25}(\alpha_s)\ \mathrm{MeV}\,,$$

Charm $\overline{\rm MS}$ quark mass determination

Weakly sensitive to long distance physics observable.

$$\langle M_B \rangle - \langle M_D \rangle = m_{b,\rm RS} - m_{c,\rm RS} + \lambda_1 \left(\frac{1}{2m_{b,\rm RS}} - \frac{1}{2m_{c,\rm RS}} \right) + O(1/m_{\rm RS}^2)\,,$$

RS scheme ($\alpha_s(M_z) = 0.118 \pm 0.003$, $N_m = 0.552 \pm 0.0552$, $\lambda_1 = 0.3 \pm 0.2$ and $m_{b,\overline{\rm MS}} = 4\,203^{+67}_{-67}$ MeV)

$$m_{c,\rm RS}(1~{\rm GeV}) = 1\,181^{+82}_{-84}(m_{b,\overline{\rm MS}})^{+4}_{-1}(\alpha_s)^{-50}_{+50}(N_m)^{-78}_{+65}(\lambda_1)~{\rm MeV};$$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,206^{+66}_{-67}(m_{b,\overline{\rm MS}})^{+1}_{-0}(\alpha_s)^{+11}_{-13}(N_m)^{-62}_{+52}(\lambda_1)~{\rm MeV},$$

RS' scheme ($m_{b,\overline{\rm MS}} = 4\,214^{+67}_{-67}$ MeV)

$$m_{c,\rm RS'}(1~{\rm GeV}) = 1\,477^{+79}_{-80}(m_{b,\overline{\rm MS}})^{+34}_{-27}(\alpha_s)^{-19}_{+19}(N_m)^{-54}_{+48}(\lambda_1)~{\rm MeV};$$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,207^{+65}_{-64}(m_{b,\overline{\rm MS}})^{-7}_{+5}(\alpha_s)^{+13}_{-14}(N_m)^{-43}_{+39}(\lambda_1)~{\rm MeV}.$$

The perturbative relation between the RS and $\overline{\rm MS}$ charm mass is convergent.

$$m_{c,\rm RS}(1~{\rm GeV}) = 1\,206 - 53 + 20 + 6 + 3 = 1\,181~{\rm MeV},$$
$$m_{c,\rm RS'}(1~{\rm GeV}) = 1\,207 + 205 + 46 + 13 + 6 = 1\,477~{\rm MeV}\,.$$

Other sources of error. (1) $\pm 40(20)$ MeV error to the RS'(RS) evaluation due to the conversion from the $\overline{\rm MS}$ to the RS'(RS) bottom quark mass. (2) $1/m^2$ terms. $\sim 15(30)$ MeV error to the RS'(RS) evaluation.

Our final result reads

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,210^{+70}_{-70}({\rm theory})^{+65}_{-65}(m_{b,\overline{\rm MS}})^{-45}_{+45}(\lambda_1)~{\rm MeV},$$

Determination of $\bar{\Lambda}$

$$\bar{\Lambda}_{\rm RS} = \langle M_B \rangle - m_{b,\rm RS} - \frac{\lambda_1}{2 m_{b,\rm RS}} + O(1/m_{b,\rm RS}^2)\,.$$

We obtain (using $m_{b,\overline{\rm MS}} = 4\,210$ MeV)

$$\bar{\Lambda}_{\rm RS}(1~\text{GeV}) = 659~\text{MeV}, \qquad \bar{\Lambda}_{\rm RS'}(1~\text{GeV}) = 401~\text{MeV}.$$

We can see that it is crucial to specify the scheme in order to give a meaningful prediction for $\bar{\Lambda}$.

The static singlet potential

Pineda (JPG)

The introduction of renormalons allows to obtain agreement between lattice simulations and perturbation theory.

$$E_s = 2m_{\rm OS} + V_{s,\rm OS} + \mathcal{O}(r^2)$$

$$E_s = 2m_{\rm RS}(\nu_f) + V_{s,\rm RS}(\nu_f) + \mathcal{O}(r^2)$$

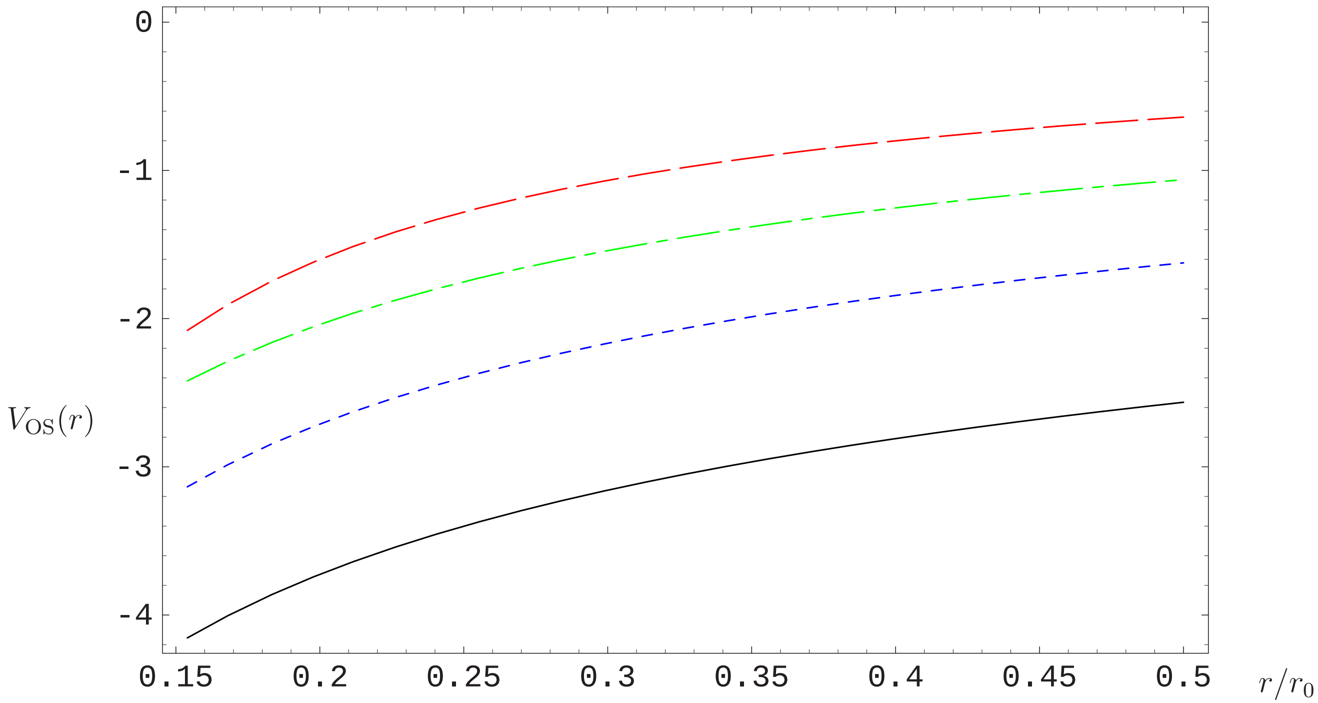


Figure 4: *Plot of $V_{\rm OS}(r)$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line). For the scale of $\alpha_s(\nu)$ ν = constant.* $\nu_{us} = 2.5\, r_0^{-1}$.

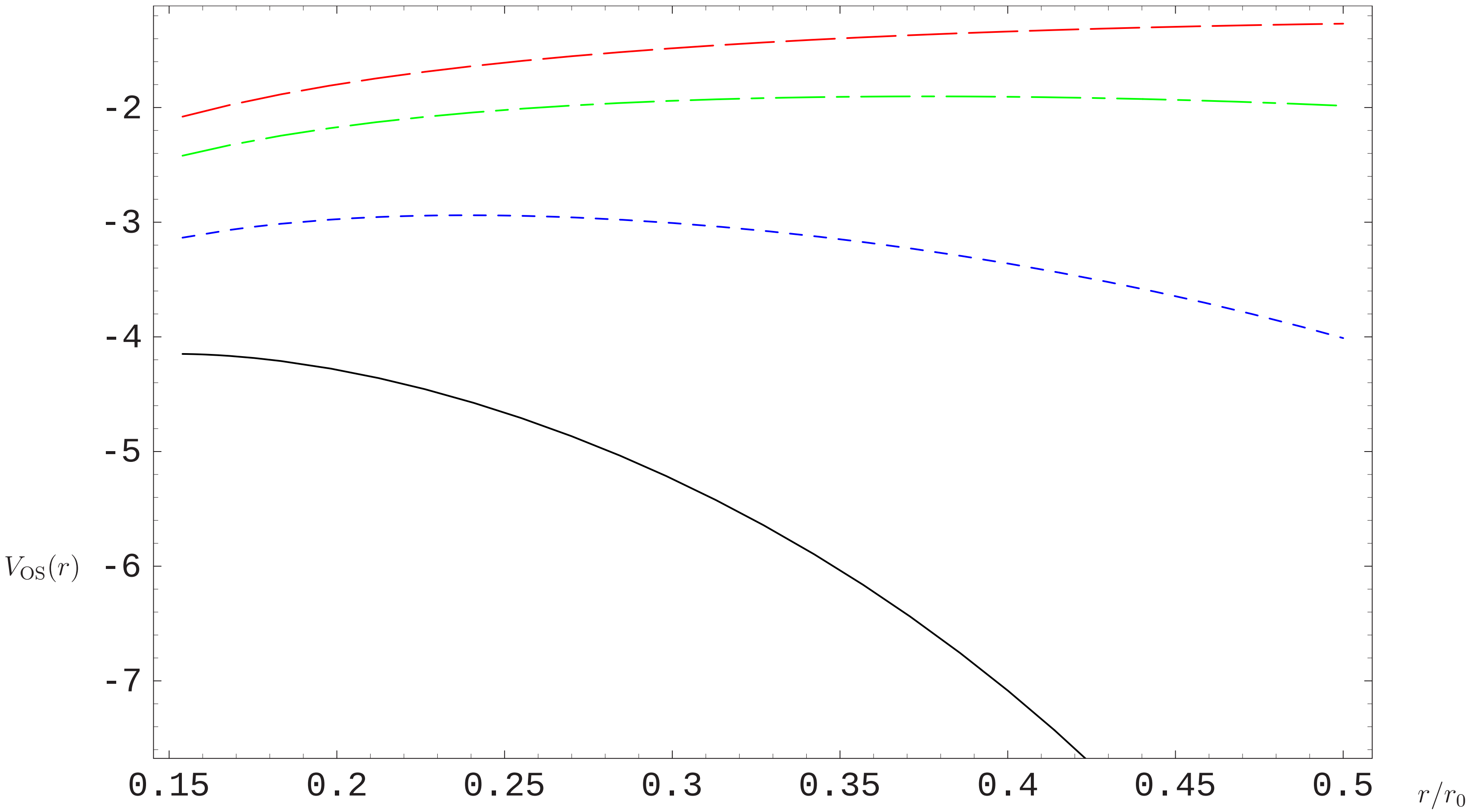


Figure 5: *Plot of* $V_{\rm OS}(r)$ *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line). For the scale of* $\alpha_s(\nu)$*, we set* $\nu = 1/r$*.* $\nu_{us} = 2.5\, r_0^{-1}$*.*

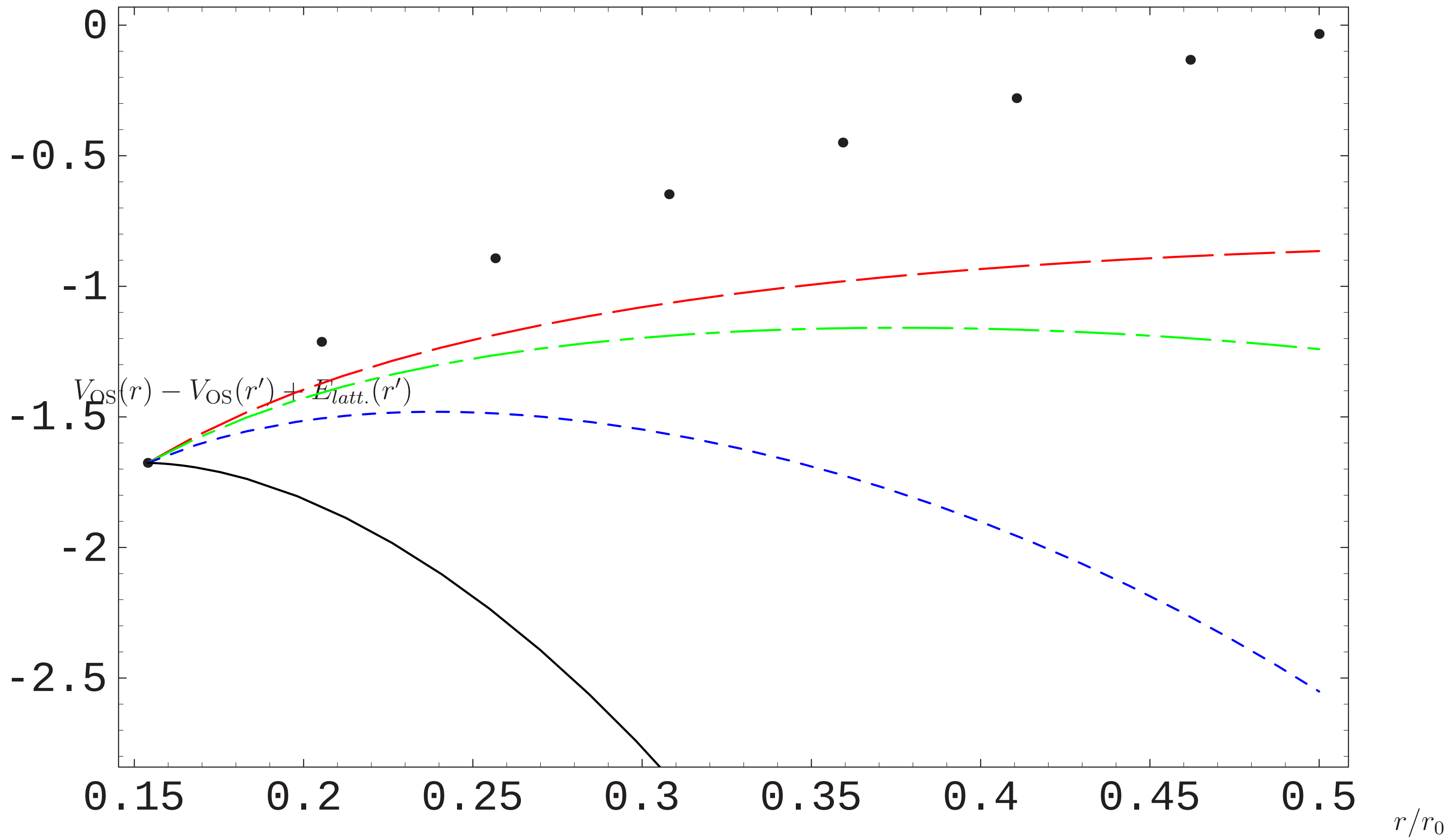


Figure 6: *Plot of* $V_{\rm OS}(r) - V_{\rm OS}(r') + E_{latt.}(r')$ *versus* r *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$*, we set* $\nu = 1/r$*.* $\nu_{us} = 2.5\, r_0^{-1}$*.*

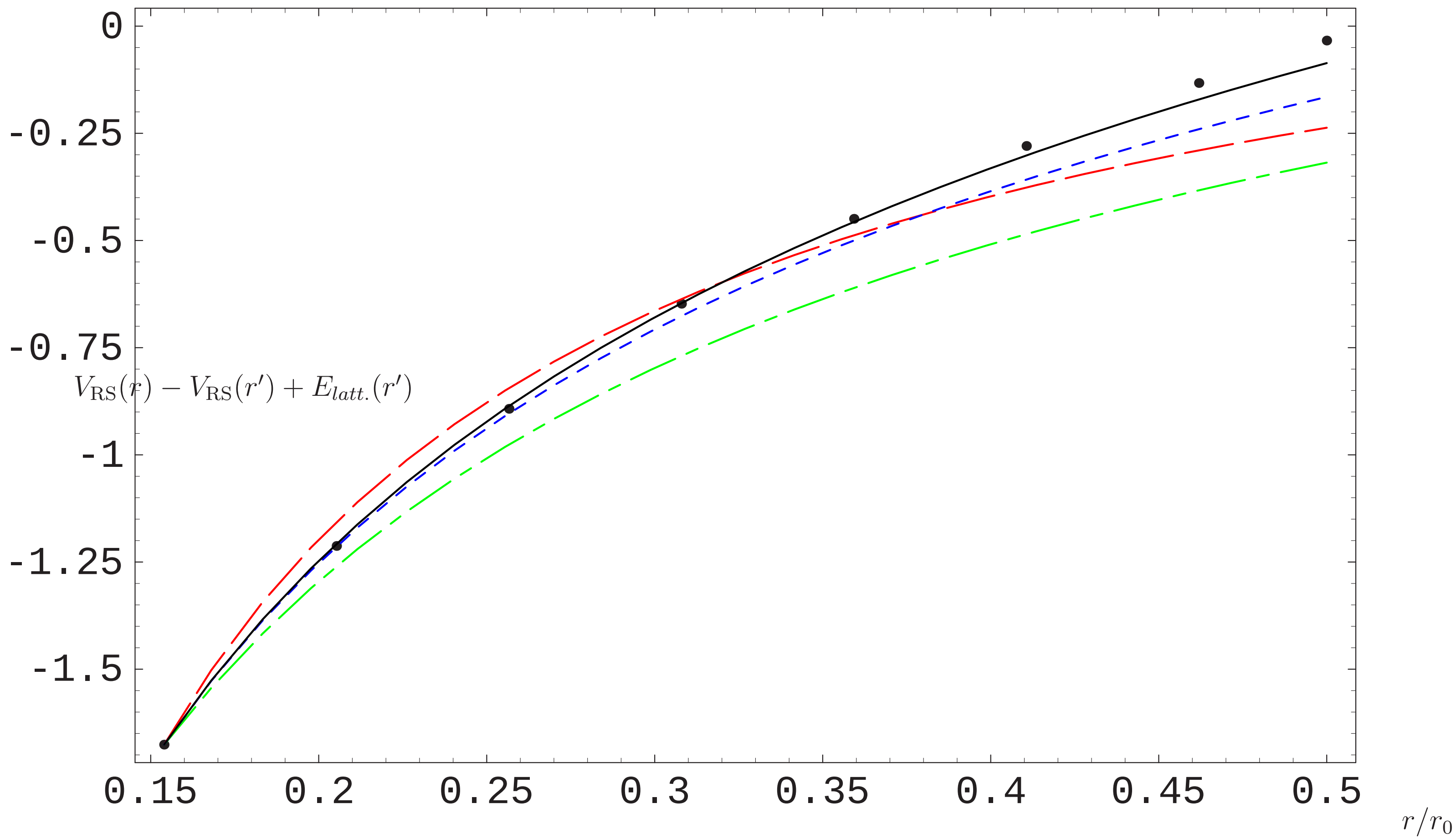


Figure 7: *Plot of* $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ *versus* r *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$ $\nu = constant$. $\nu_{us} = 2.5\, r_0^{-1}$.

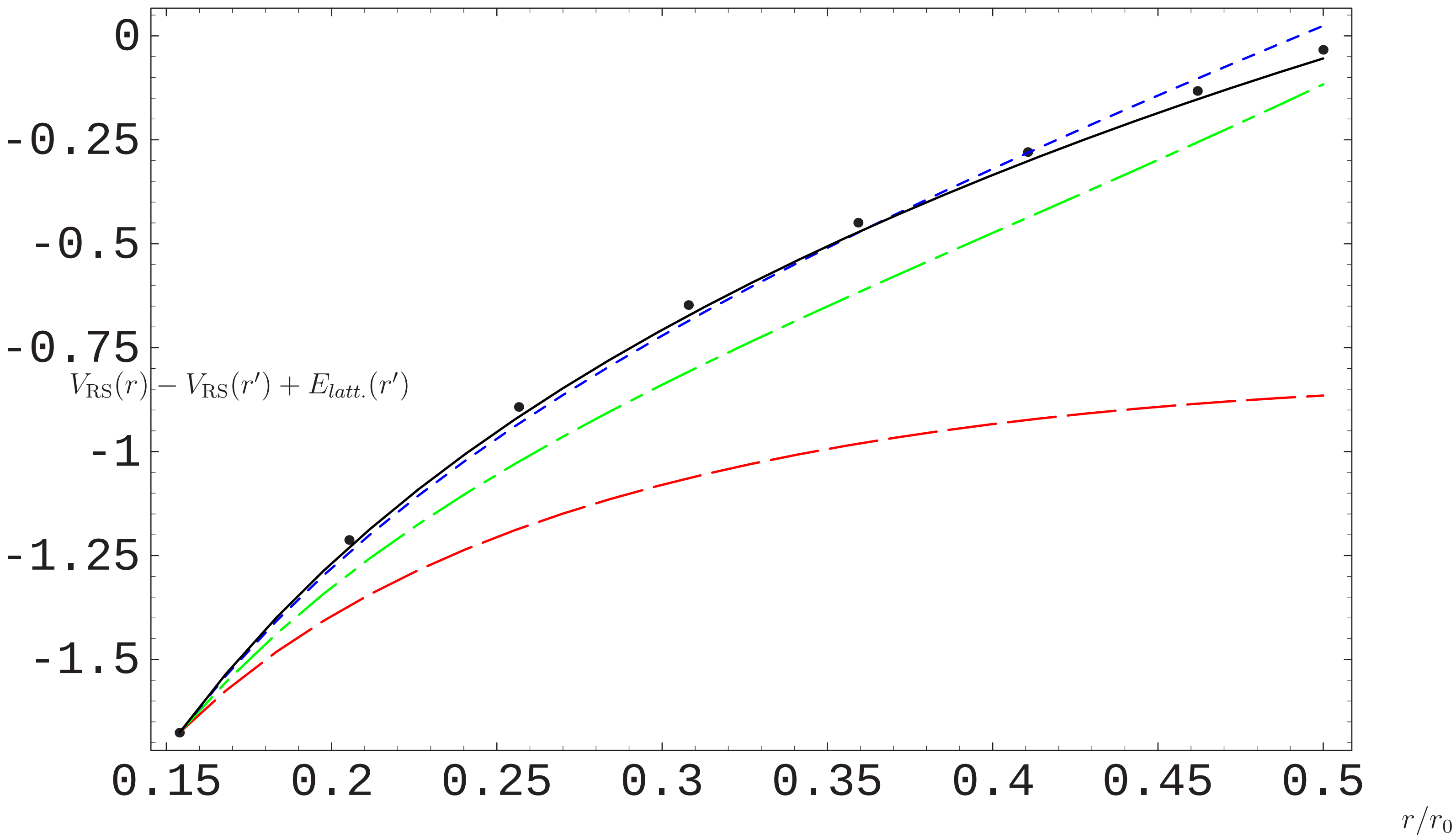


Figure 8: *Plot of* $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ *versus* r *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$*, we set* $\nu = 1/r$*.* $\nu_f = \nu_{us} = 2.5\, r_0^{-1}$*.*

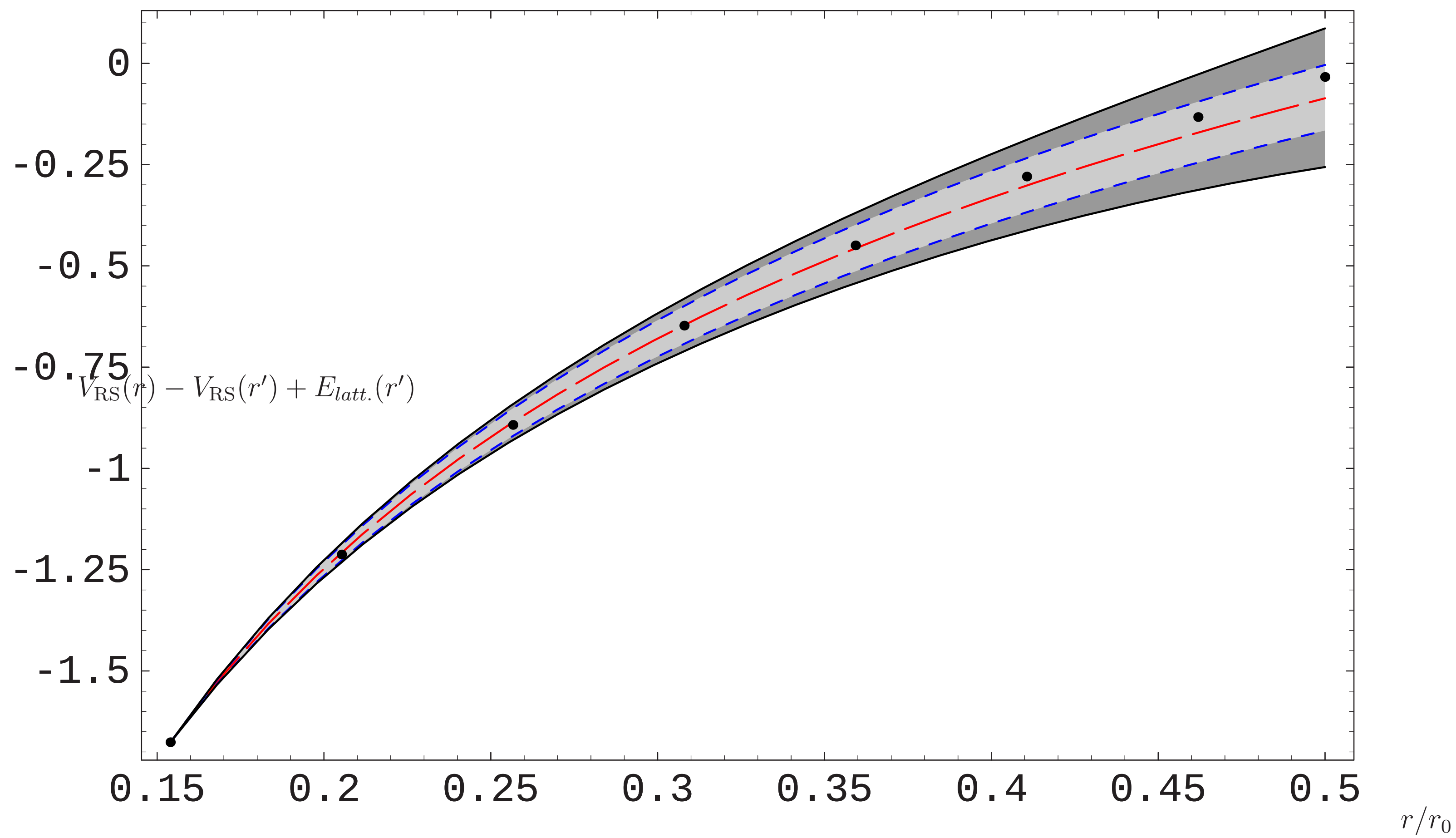


Figure 9: *Plot of* $V_{\rm RS}(r) - V_{\rm RS}(r') + E_{latt.}(r')$ *versus* r *at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of* $\alpha_s(\nu)$ $\nu = constant$. $\nu_{us} = 2.5\, r_0^{-1}$.

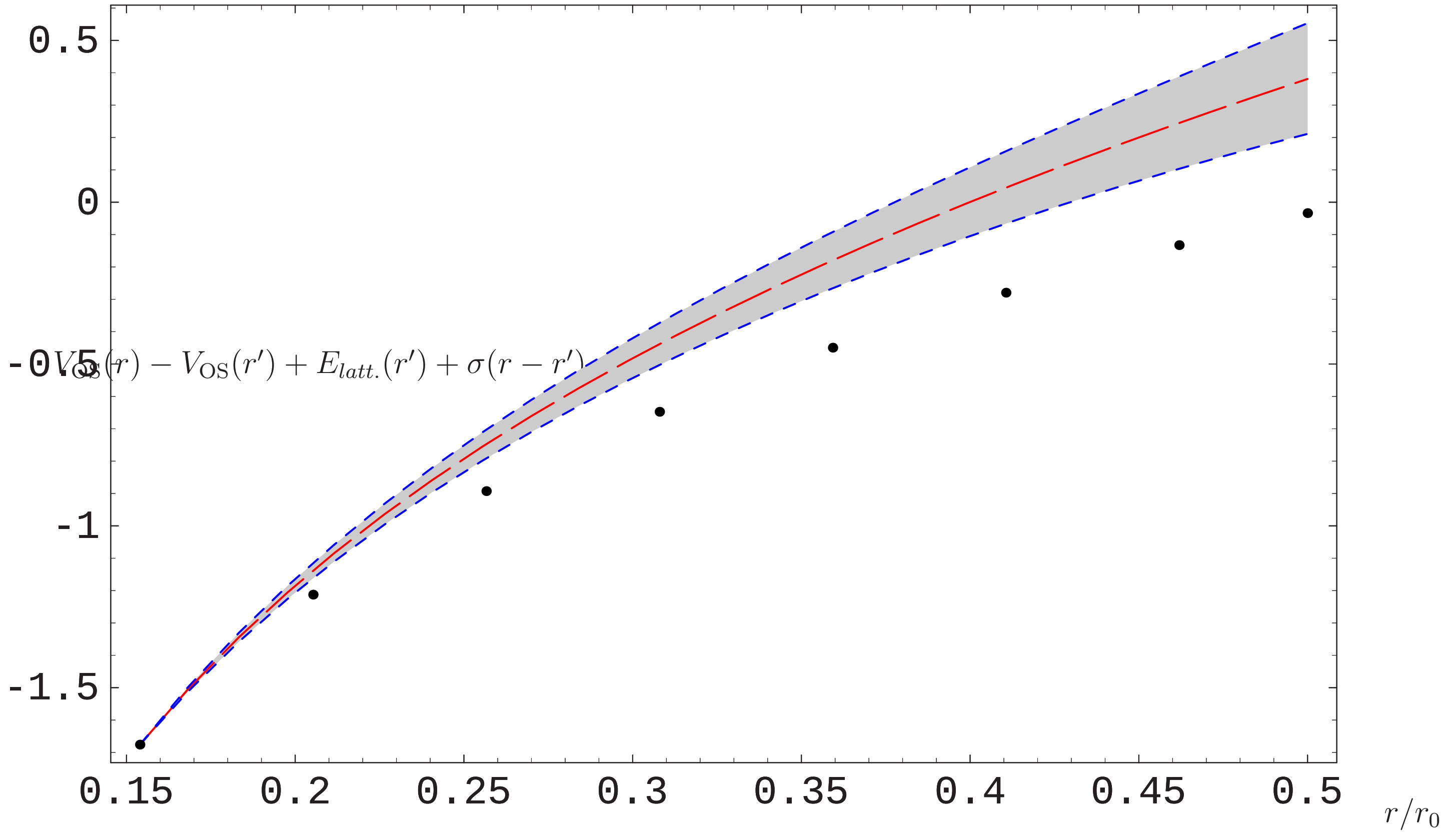


Figure 10: *Plot of $V_{\rm OS}(r) - V_{\rm OS}(r') + E_{latt.}(r') + \sigma(r-r')$ versus r at three loops (estimate) with the leading ultrasoft log compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu =$ constant. $\sigma = 1.35\, r_0^{-2}$ and $\nu_{us} = 2.5\, r_0^{-1}$.*

Constraint on the size of nonperturbative effecs for heavy quarkonium.

No linear non-perturbative potential at short distances.

pNRQCD: the static limit $O(1/m^0)$

Brambilla, Pineda, Soto, Vairo (NPB); Bali, Pineda (PRD)

Extra (approximate) symmetries in the effective theory.

NRQCD: $D_{\infty h}$ (substituting parity by CP).

pNRQCD: $O(3) \times$C. Softly broken by the multipole expansion.

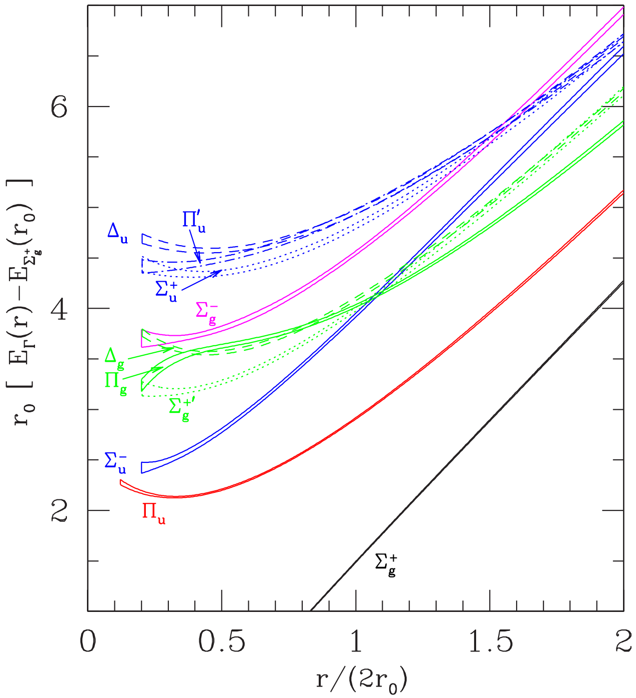


Figure 11: *Energies for different gluonic excitation between static quarks. Juge et al. hep-lat/9809015,* $r_0 \simeq 0.5$ *fm.*

Degeneracies:

$$\Sigma_g^{+\prime} \sim \Pi_g \; ; \qquad \Sigma_g^- \sim \Pi'_g \sim \Delta_g \; ;$$
$$\Sigma_u^- \sim \Pi_u \; ; \qquad \Sigma_u^+ \sim \Pi'_u \sim \Delta_u \, .$$

Gluelumps $O^a H^a$	$L=1$	$L=2$
$\Sigma_g^{+\prime}$	$\mathbf{r}\cdot\mathbf{E}\,,\mathbf{r}\cdot(\mathbf{D}\times\mathbf{B})$	
Σ_g^-		$(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{B})$
Π_g	$\mathbf{r}\times\mathbf{E}\,,\mathbf{r}\times(\mathbf{D}\times\mathbf{B})$	
Π_g'		$\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{B}+\mathbf{D}(\mathbf{r}\cdot\mathbf{B}))$
Δ_g		$(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{B})^j+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{B})^i$
Σ_u^+		$(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{E})$
Σ_u^-	$\mathbf{r}\cdot\mathbf{B}\,,\mathbf{r}\cdot(\mathbf{D}\times\mathbf{E})$	
Π_u	$\mathbf{r}\times\mathbf{B}\,,\mathbf{r}\times(\mathbf{D}\times\mathbf{E})$	
Π_u'		$\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{E}+\mathbf{D}(\mathbf{r}\cdot\mathbf{E}))$
Δ_u		$(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{E})^j+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{E})^i$

Table 6: *Operators H for the Σ, Π and Δ gluonic excitations between static quarks in pNRQCD up to dimensions 3. The covariant derivative is understood in the adjoint representation. $\mathbf{D}\cdot\mathbf{B}$ and $\mathbf{D}\cdot\mathbf{E}$ do not appear, the first because it is identically zero after using the Jacobi identity, while the second gives vanishing contributions after using the equations of motion.*

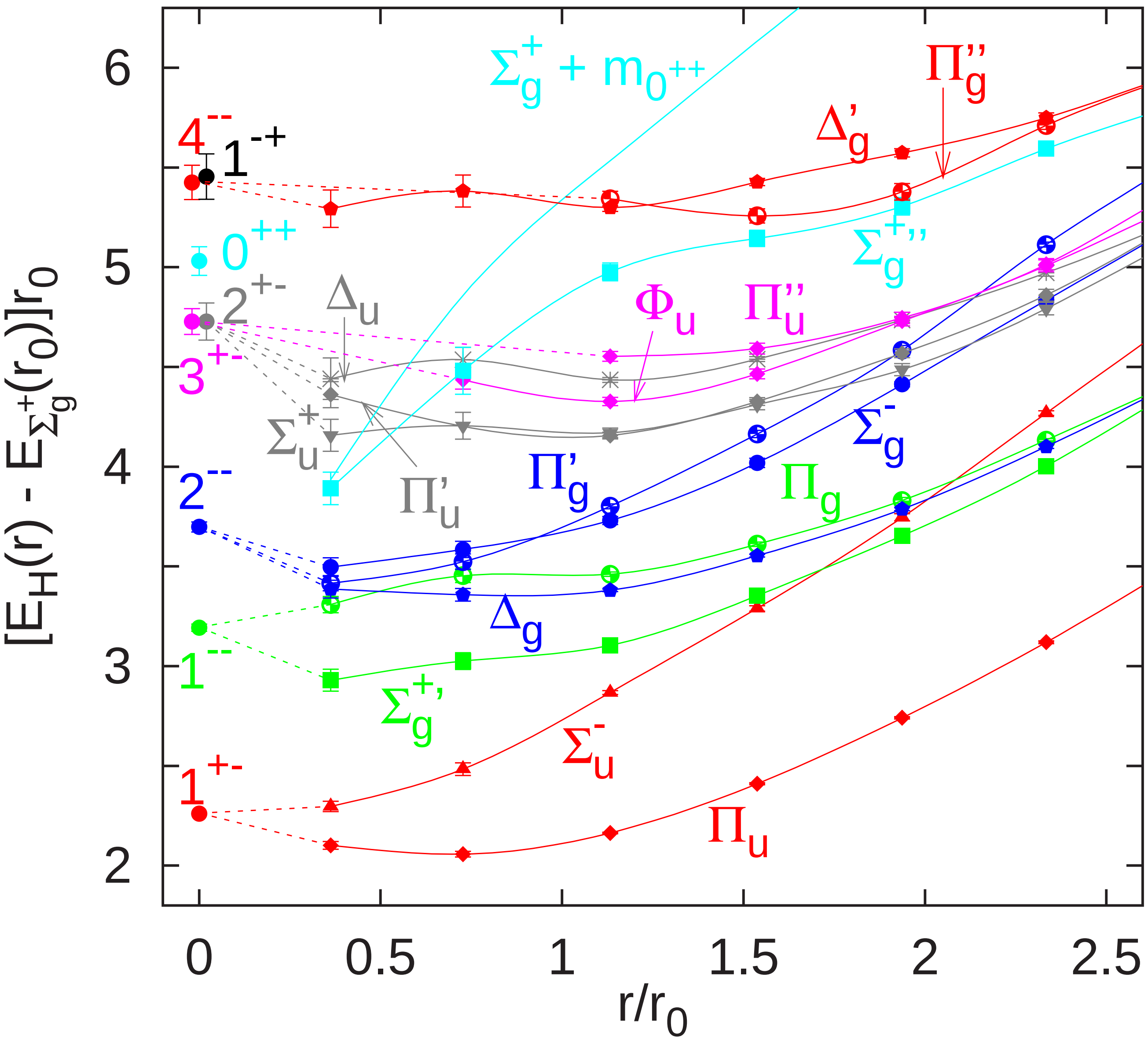


Figure 12: Different hybrid potentials Juge et al. 2003 at a lattice spacing $a_\sigma \approx 0.2$ fm $\approx 0.4\,r_0$, where $r_0 \approx 0.5$ fm, in comparison with the gluelump spectrum, extrapolated to the continuum limit Foster and Michael, 1999 (circles, left-most data points). The gluelump spectrum has been shifted by an arbitrary constant to adjust the 1^{+-} state with the Π_u and Σ_u^- potentials at short distance. In addition, we include the sum of the ground state (Σ_g^+) potential and the scalar glueball mass $m_{0^{++}}$ Bali et al., 1993; Lucini and Teper, 2001; Morningstar and Peardon, 1999. The lines are drawn to guide the eye.

Predictions of the shape of the static energies:

We compute ($H = O^a H^a$)

$$\langle 0|H(\mathbf{R},\mathbf{r},T/2)H^\dagger(\mathbf{R}',\mathbf{r}',-T/2)|0\rangle$$
$$\sim \delta^3(\mathbf{R}-\mathbf{R}')\delta^3(\mathbf{r}-\mathbf{r}')\, e^{-iTV_H(r)}$$

for large T. At leading order in the multipole expansion we obtain

$$V_H(r) = V_o(r) + \frac{i}{T}\ln\langle H^a(T/2)\phi(T/2,-T/2)^{\rm adj}_{ab}H^b(-T/2)\rangle,$$

where the $T \to \infty$ limit is understood. The general structure of the gluonic correlator is the following

$$\langle H^a(T/2)\phi(T/2,-T/2)^{\rm adj}_{ab}H^b(-T/2)\rangle^{\rm nonpert.}$$
$$\simeq h\, e^{-i\Lambda_H T} + h'\, e^{-i\Lambda'_H T} + \dots .$$

Since we are in the static limit, $1/T \ll \Lambda_{\rm QCD} \sim \Lambda_{\rm H} < \Lambda'_{\rm H} < \dots$, one can approximate the right-hand side by just keeping the first exponential. Then we get at leading order in the multipole expansion

$$\boxed{V_H(r) = V_o(r) + \Lambda_H}$$

We can relate, in the short-distance limit, the behavior of the energies for the gluonic excitations between static quarks with the large time behavior of some gluonic correlators, in particular with their correlation length.

$$\langle 0|F^a_{\mu\nu}(t)\phi(t,0)^{\rm adj}_{ab}F^b_{\mu\nu}(0)|0\rangle\,.$$

One can parameterize this correlator as a function of two scalar functions:

$$\langle 0|\mathbf{E}^a(t)\phi(t,0)^{\rm adj}_{ab}\mathbf{E}^b(0)|0\rangle \quad \text{and} \quad \langle 0|\mathbf{B}^a(t)\phi(t,0)^{\rm adj}_{ab}\mathbf{B}^b(0)|0\rangle\,,$$

with correlation lengths: $T_E = 1/\Lambda_E$ and $T_B = 1/\Lambda_B$ respectively. We can conclude that

$$\boxed{T_E < T_B}$$

One could eventually go beyond by computing corrections in the multipole expansion.

$$E_H = 2m_{\rm OS} + V_{o,{\rm OS}} + \Lambda_H^{\rm OS} + \mathcal{O}(r^2)$$

$$E_H = 2m_{\rm RS}(\nu_f) + V_{o,{\rm RS}}(\nu_f) + \Lambda_H^{\rm RS}(\nu_f) + \mathcal{O}(r^2)$$

$$E_H = 2m_{\rm latt}(1/a) + V_{o,{\rm latt}}(1/a) + \Lambda_H^{\rm latt}(1/a) + \mathcal{O}(r^2)$$

$$V_o = \frac{1}{2N_c}\frac{\alpha_s}{r} + \cdots ({\rm two-loops: Kniehl, Penin, Schroeder, Smirnov, Steinhauser})$$

$$2N_m + N_{V_o} + N_\Lambda = 0$$

Factorization scale $\nu_f \leftrightarrow 1/a$; $\delta m_{\rm RS}(\nu_f)$, $\delta\Lambda_{\rm RS}(\nu_f)$; $\delta m_{\rm latt}(1/a)$, $\delta\Lambda_{\rm latt}(1/a)$.

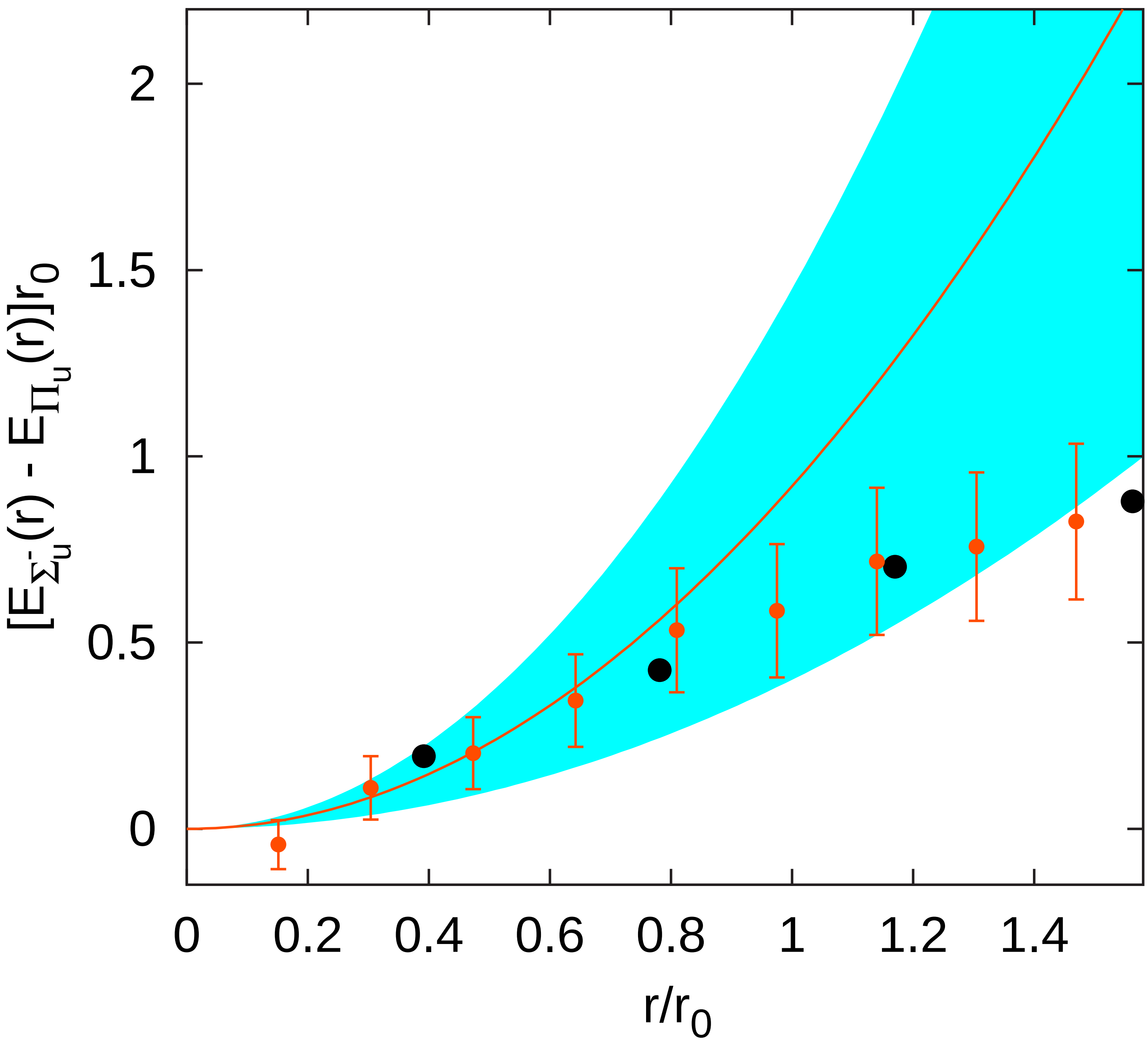


Figure 13: Splitting between the Σ_u^- and the Π_u potentials, extrapolated to the continuum limit, and the comparison with a quadratic fit to the $r \lesssim 0.5\, r_0$ data points ($r_0^{-1} \approx 0.4$ GeV). The big circles correspond to the data of Juge et al., obtained at finite lattice spacing $a_\sigma \approx 0.39\, r_0$. The errors in this case are smaller than the symbols.

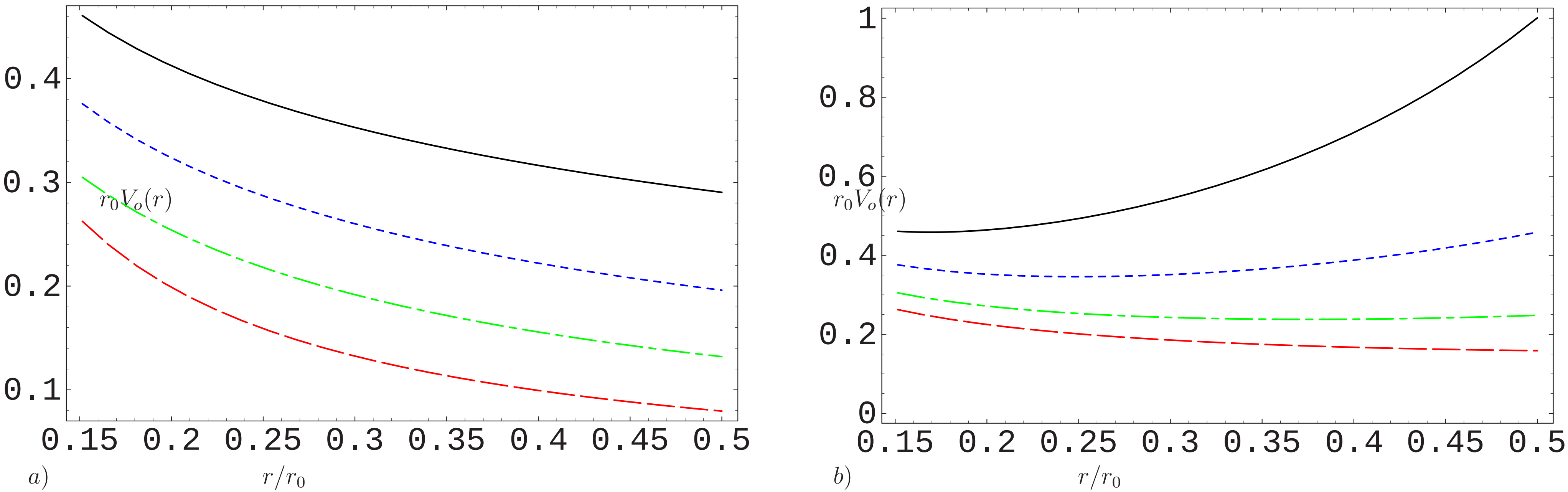


Figure 14: $r_0V_o(r)$ *(the octet potential in the OS scheme) at tree level (dashed lines), one-loop (dashed-dotted lines), two loops (dotted lines) and three loops (estimate) plus the leading single ultrasoft log (solid lines). Fig. a) corresponds to the scale* $\nu = \nu_i$ *and Fig. b) to* $\nu = 1/r$*. In both cases,* $\nu_{us} = 2.5\, r_0^{-1}$*. Only the solid curves depend on this choice.*

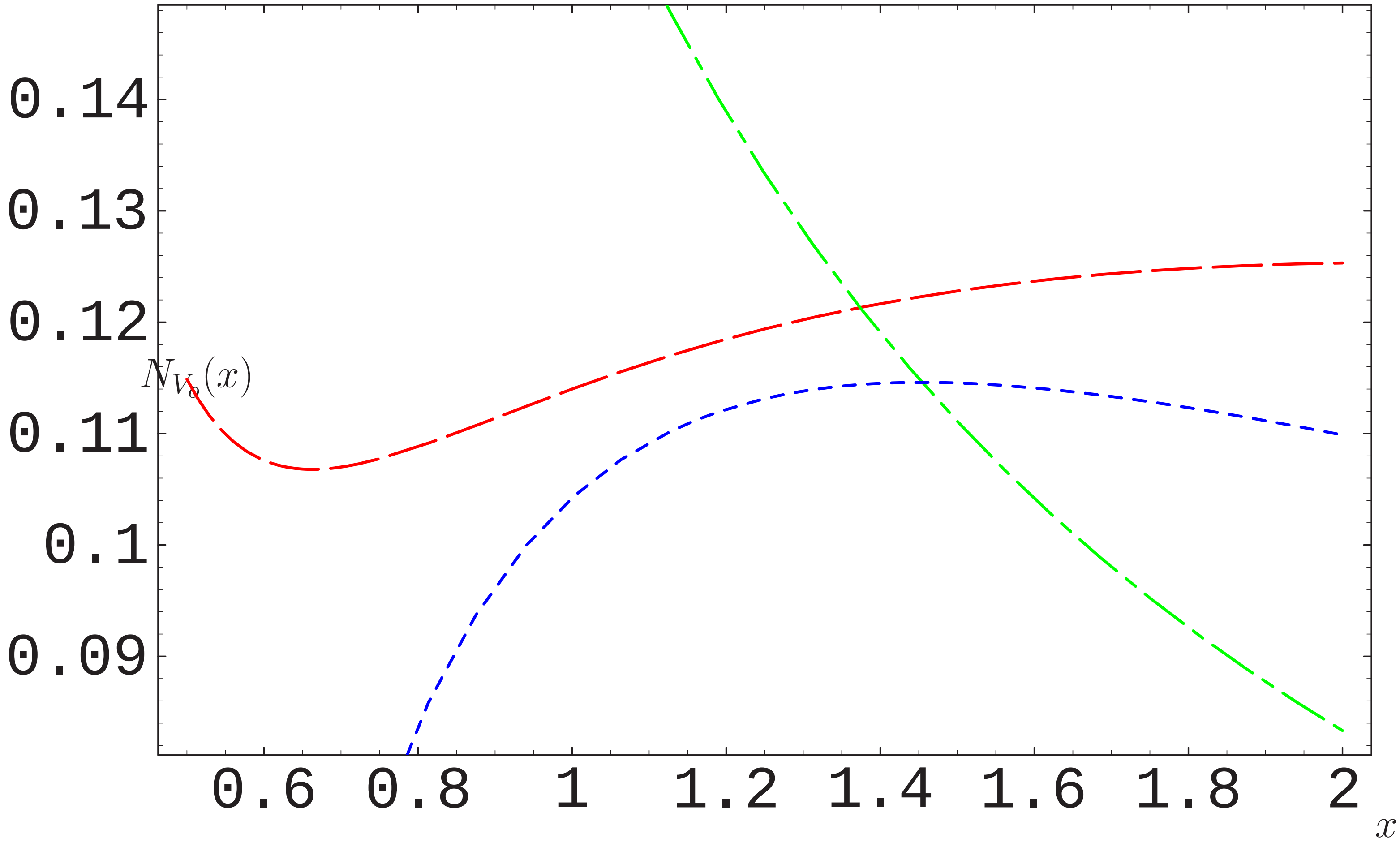


Figure 15: $x \equiv \nu r$ *dependence of* N_{V_o} *for* $n_f = 0$ *at LO (dashed-dotted line), NLO (dotted line) and NNLO (dashed line).*

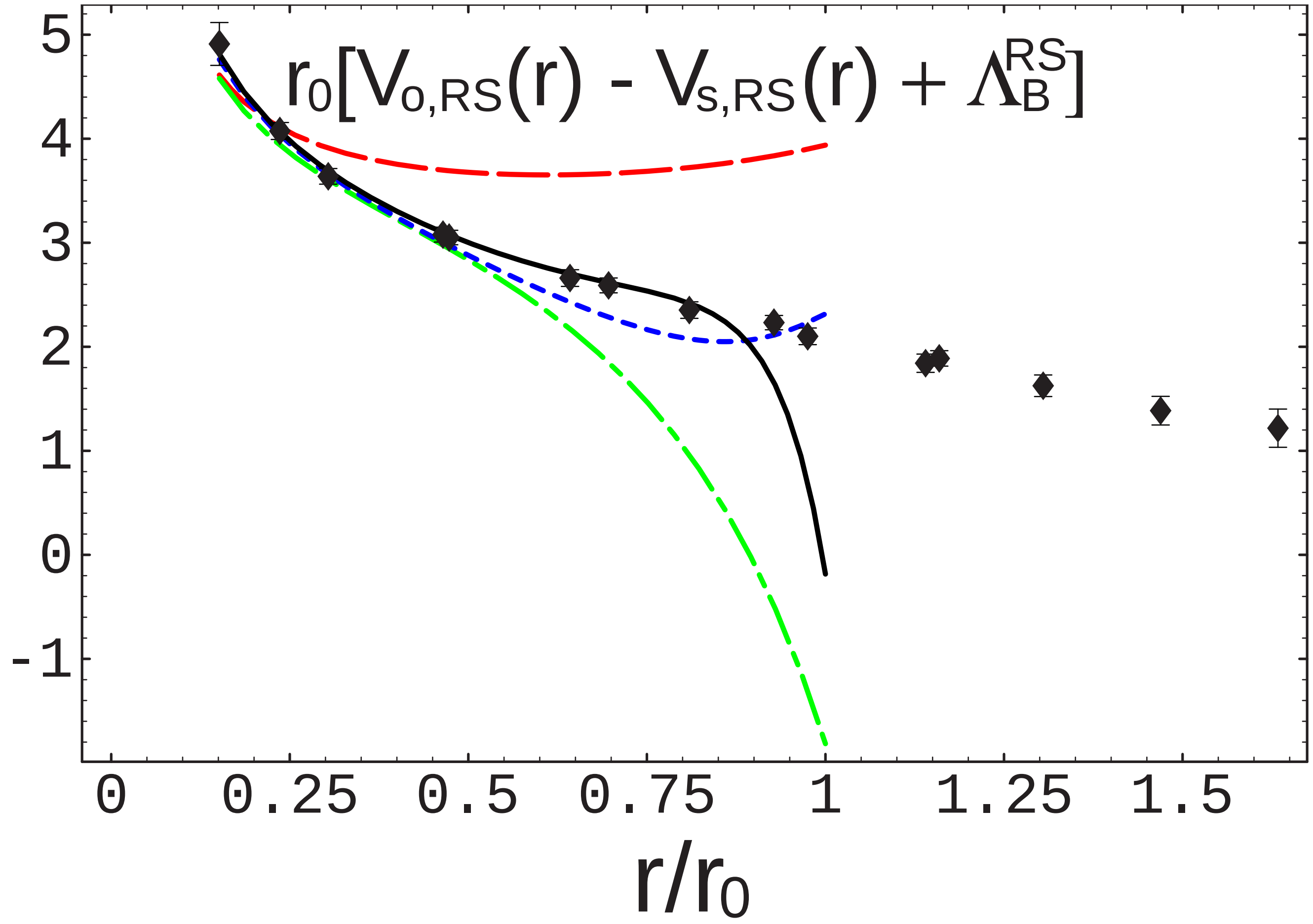


Figure 16: Splitting between the Π_u and the Σ_g^+ potentials and the comparison with the theoretical prediction with $\nu = 1/r$ for $\nu_f = 2.5\, r_0^{-1}$. $r_0[(V_{o,\rm RS} - V_{s,\rm RS})(r) + \Lambda_B^{\rm RS}]$ is plotted versus r at tree level (dashed line), one-loop (dashed-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line).

Table 7: Absolute values for the gluelump masses in the continuum limit in the RS scheme at $\nu_f = 2.5\, r_0^{-1} \approx 1$ GeV, in r_0 units and in GeV. Note that an additional uncertainty of about 10 % should be added to the last column to account for the quenched approximation. We also display examples of creation operators H for these states. The curly braces denote complete symmetrization of the indices.

J^{PC}	H	$\Lambda_H^{\rm RS} r_0$	$\Lambda_H^{\rm RS}/\mathrm{GeV}$
1^{+-}	B_i	2.25(39)	0.87(15)
1^{--}	E_i	3.18(41)	1.25(16)
2^{--}	$D_{\{i}B_{j\}}$	3.69(42)	1.45(17)
2^{+-}	$D_{\{i}E_{j\}}$	4.72(48)	1.86(19)
3^{+-}	$D_{\{i}D_jB_{k\}}$	4.72(45)	1.86(18)
0^{++}	$\mathbf{B}^2$	5.02(46)	1.98(18)
4^{--}	$D_{\{i}D_jD_kB_{l\}}$	5.41(46)	2.13(18)
1^{-+}	$(\mathbf{B}\wedge\mathbf{E})_i$	5.45(51)	2.15(20)

Heavy-light systems

$$M_B = m_{\rm RS}(\nu_f) + \bar{\Lambda}^{\rm RS}(\nu_f) + \mathcal{O}(1/m)$$

$$M_B = m_{\rm latt}(1/a) + \bar{\Lambda}^{\rm latt}(1/a) + \mathcal{O}(1/m)$$

$$\bar{\Lambda}^{\rm RS}(2.5\, r_0^{-1}) = [1.17 \pm 0.08({\rm latt.}) \pm 0.13({\rm th.}) \pm 0.09(\Lambda_{\overline{\rm MS}})]\, r_0^{-1}\,.$$

$$m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}}) = [4191 \pm 29({\rm latt.}) \pm 47({\rm th.}) \pm 1(\Lambda_{\overline{\rm MS}})]\,{\rm MeV}\,,$$

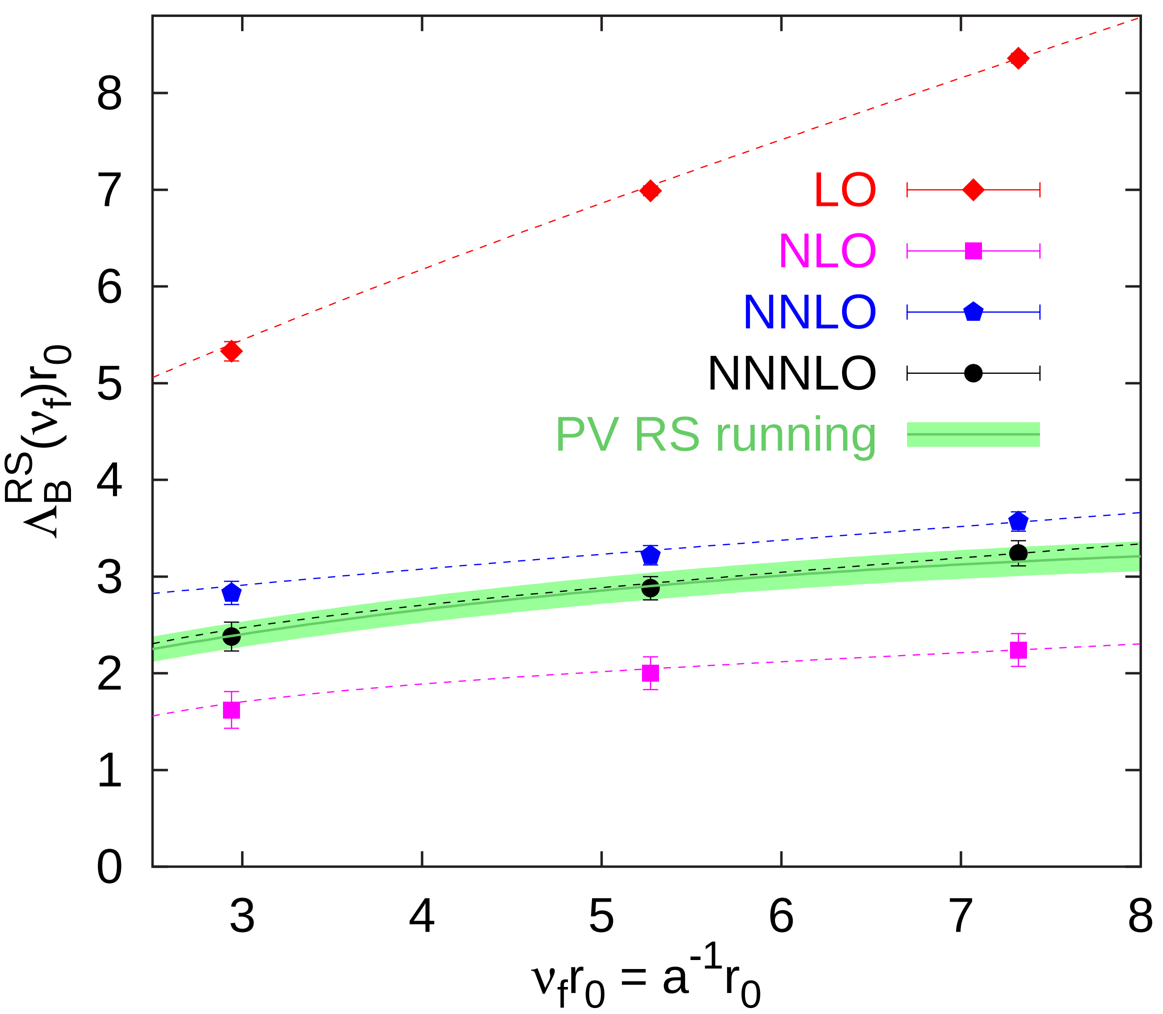


Figure 17: *The lowest gluelump mass Λ_B^L as obtained on the lattice (diamonds), as well as converted into the RS scheme at NLO (squares), NNLO (pentagons) and NNNLO* (NNNLO estimate, circles). The error band corresponds to the result for $\Lambda_B^{\rm RS}$ without the "theoretical" error, run to different scales, according to the PV prescription.*

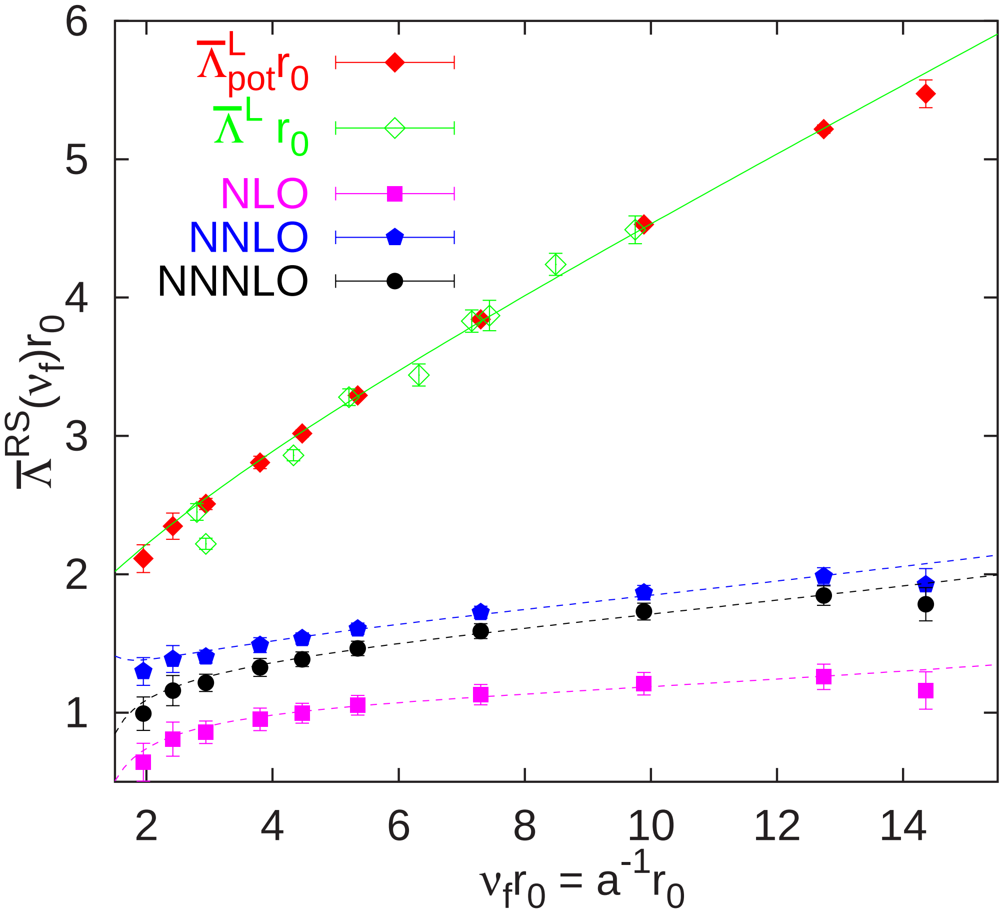


Figure 18: The binding energy $\overline{\Lambda}^{L}_{\rm pot}$, in the lattice scheme (full diamonds), in comparison with $\overline{\Lambda}^{L}$ (open diamonds). The constant Δ has been adjusted by requiring agreement between the two data sets at $r_0 \approx 7.3\,a$. The uncertainty of $\Delta = (0.988 \pm 0.067)\, r_0^{-1}$ is not included into the errors. NLO, NNLO and NNNLO refer to transformations of $\overline{\Lambda}^{L}_{\rm pot}$ into the RS scheme to different orders in perturbation theory. The solid line corresponds to the NNNLO expectation with $\Lambda_{\overline{\rm MS}} \approx 0.602\, r_0^{-1}$, and the central value, $\overline{\Lambda}^{\rm RS}(\nu_f = 9.76\, r_0^{-1}) = 1.70\, r_0^{-1}$.

Conclusions

- Complete factorization of scales within dimensional regularization for nonrelativistic systems (by using effective field theories: HQET, pNRQCD, ...).
- Renormalon free working scheme preserving the power counting rules.
  - Calculation of the normalization constant of the first renormalon of the pole mass and static potential.

$$N_m = 0.55 \qquad N_{V_s} = -1.08 \ (n_f = 4)$$

  - Estimates of the higher order coefficients of the perturbative series relating the pole mass with the $\overline{\rm MS}$ mass (and the singlet static potential with α_s) have been obtained without relying on the large β_0 approximation.
  - Determination of the $\overline{\rm MS}$ bottom and charm masses.

$$m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}}) = 4\,210^{+90}_{-90}({\rm theory})^{-25}_{+25}(\alpha_s) \ {\rm MeV},$$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,210^{+70}_{-70}({\rm theory})^{+65}_{-65}(m_{b,\overline{\rm MS}})^{-45}_{+45}(\lambda_1) \ {\rm MeV}.$$

  - Good description of the (lattice) static singlet and octet potential at short distances with perturbation theory.
  - Determination of the gluelump masses, $\bar{\Lambda}$ and m_b mass from lattice.

Another possibility: to perform directly the matching to singlet and octet fields, one naturally ends up with Wilson loops (Brambilla-Vairo-Soto-Pineda).

Expansion in $1/M$. HQET can be used in the matching (static sources).

A) Wilson loop formalism. Suitable if we can not work perturbatively. For example, the computation of the following Green function in both theories

$$\langle 0|Q_2^\dagger(x_2)\phi(x_2,x_1)Q_1(x_1)Q_1^\dagger(y_1)\phi(y_1,y_2)Q_2(y_2)|0\rangle,$$

NRQCD

$$\delta^3(\mathbf{x}_1-\mathbf{y}_1)\delta^3(\mathbf{x}_2-\mathbf{y}_2)\langle W_\Box\rangle,$$

pNRQCD

$$Z_s(\mathbf{r})\delta^3(\mathbf{x}_1-\mathbf{y}_1)\delta^3(\mathbf{x}_2-\mathbf{y}_2)e^{-iTV_s^{(0)}(\mathbf{r})}$$

One obtains:

$$V^{(0)}(\mathbf{r})=\lim_{T\to\infty}\frac{i}{T}\log\langle W_\Box\rangle=-C_f\frac{\alpha_s}{r}+O(\alpha_s^2)\quad \text{Wilson, Susskind}$$

$$V^{(1,0)}=-\lim_{T\to\infty}\int_0^T dt\,\frac{t}{2}\,\langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\rangle_c=-C_fC_A\frac{\alpha_s^2}{4r^2}+O(\alpha_s^3)\ \text{Brambilla, Soto, Vairo, P.}$$

and $O(1/m^2)$ ...

B) Even another method to get the potentials in terms of Wilson loops. Comparison between states and matrix elements in NRQCD and pNRQCD starting from the static solution. Closer in philosophy, and to some extent in the procedure, to standard quantum mechanics perturbation theory.

No complete/correct method to obtain the subleading potentials in $1/m$ in the past.

$V_{LS}^{(2,0)}$, $V_{S^2}^{(1,1)}$, $V_{S_{12}}^{(1,1)}$: Eichten, Feinberg; Peskin; Gromes; Cheng, Kuang, Oakes; Barchielli, Montaldi, Prosperi; Barchielli, Brambilla, Prosperi; Szczepaniak, Swanson; Vairo, Pineda

$V_{L_1S_2}^{(1,1)}$: Vairo, Pineda; Szczepaniak, Swanson

$V_{\mathbf{p}^2}^{(2,0)}$, $V_{\mathbf{L}^2}^{(2,0)}$, $V_{\mathbf{p}^2}^{(1,1)}$, $V_{\mathbf{L}^2}^{(1,1)}$: Barchielli, Montaldi, Prosperi; Barchielli, Brambilla, Prosperi; Vairo, Pineda

$V^{(1,1)}$: Brambilla, Soto, Vairo, Pineda

$V_r^{(2,0)}$, $V_r^{(1,1)}$: Vairo, Pineda

Check. Calculation in the OS scheme + renormalon cancellation ($\nu = m_{c,\overline{\rm MS}}$). $m_{b,\overline{\rm MS}} = 4\,210 \pm 90$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,254^{+85}_{-84}(m_{b,\overline{\rm MS}})^{+17}_{-12}(\alpha_s)^{-49}_{+45}(\lambda_1)\ {\rm MeV}\,.$$

$$m_{b,{\rm OS}} - m_{c,{\rm OS}} = 2\,956 + 490 - 14 - 32 + 22 = 3\,423\ {\rm MeV}\,.$$

Potential large logs. With $\nu = 2m_{c,\overline{\rm MS}}$, we obtain

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,239{\rm MeV}$$

and the expansion seems to improve:

$$m_{b,{\rm OS}} - m_{c,{\rm OS}} = 2\,971 + 345 + 79 + 19 + 10 = 3\,424\ {\rm MeV}.$$