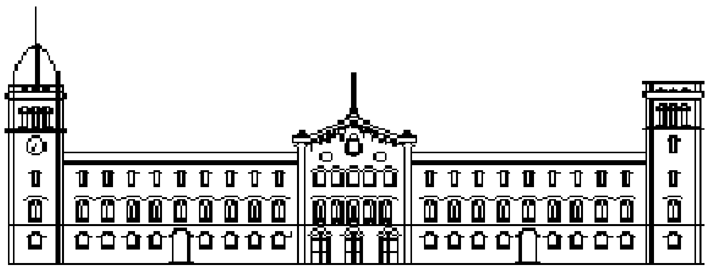
RENORMALIZATION GROUP IN

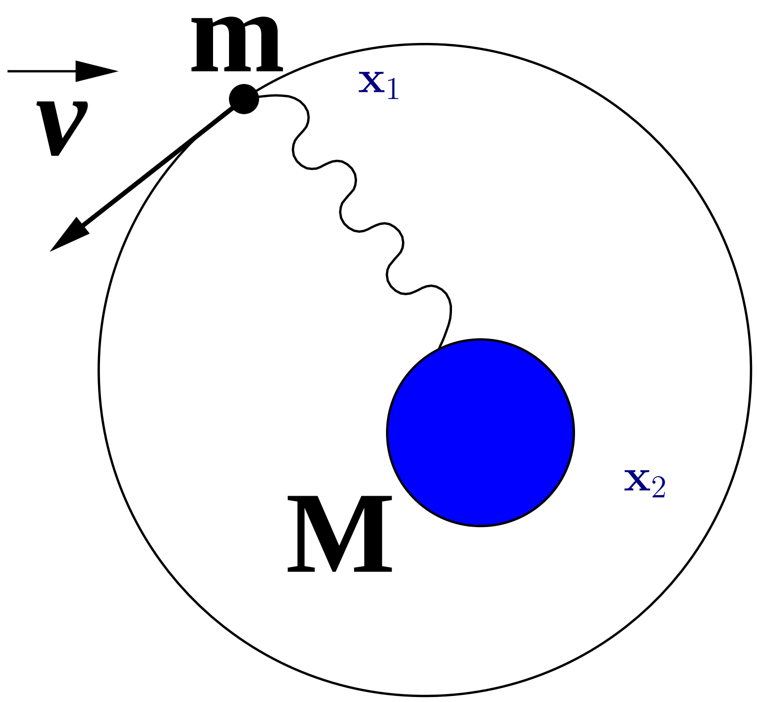
NON-RELATIVISTIC EFFECTIVE FIELD THEORIES

ANTONIO PINEDA

(ECM, U. Barcelona)



Hydrogen atom



$$V(r) = -\frac{Z_1 Z_2 \alpha}{r}$$

$$v \sim \alpha \ll 1$$

$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \qquad \mathbf{X} = \frac{m}{m+M}\mathbf{x}_1 + \frac{M}{m+M}\mathbf{x}_2$$

Scales: m, $m\alpha$, $m\alpha^2$, ...

NR Effective Field Theories

Our aim is to provide a systematic method to deal with NR bound state systems. We will introduce a hierarchy of EFTs when sequentially integrating out each scale (only one scale in each step, strong simplification).

QCD

↓ Integrating out the hard scale (m)

NRQCD — Caswell, Lepage

↓ Integrating out the soft scale (mv)

pNRQCD — Soto, Pineda

$$\left.\begin{array}{l}\left(i\partial_0 - \dfrac{\mathbf{p}^2}{2m} - V_0(r)\right)\Phi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with other low} \\ \quad \text{energy degrees of freedom}\end{array}\right\} \text{potential NRQCD} \qquad E \sim mv^2$$

In the perturbative case the starting point is $V_0 = -C_f \dfrac{\alpha}{r}$.

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Resummation of Logarithms in Deep Inelastic scattering
Resummation of logarithms in HQET
Electroweak logarithms
and so on ...

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BUT!!!

WHAT ABOUT THE FIRST QUANTUM-FIELD-THEORY LOG?

THE LAMB SHIFT

$$\delta E \sim m\alpha^4 + m\alpha^5 \ln\alpha + (???) m\alpha^6 \ln^2\alpha + \cdots$$

Summing logs in non-relativistic systems

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\Gamma(V_Q(nS) \to e^+e^-) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \\
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Relevant for...

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Relevant for...
Determination of the bottom and charm masses

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Decays of heavy quarkonium
Heavy quarkonium sum rules
t-$\bar{t}$ production near threshold: m_t, α_s, Higgs-top coupling

Renormalization group in NRQCD (LL) (Soft running)

Aim: to obtain the running of the NRQCD matching coefficients: $\left(\alpha_s \ln \frac{m}{\nu}\right)^n$

Relevant for:

- pNRQCD in the perturbative regime
- pNRQCD in the nonperturbative regime
- "Standard" NRQCD

$$\mathcal{L}_{NRQCD} = \bar{\Psi} i\gamma^0 D_0 \Psi + \bar{\Psi}\{\frac{\mathbf{D}^2}{2m} + c_F\, g\frac{\mathbf{\Sigma}\cdot\mathbf{B}}{2m} + c_D\, g\frac{\gamma^0\left(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D}\right)}{8m^2}$$
$$+ic_S\, g\frac{\gamma^0\mathbf{\Sigma}\cdot\left(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}\right)}{8m^2} + \frac{\mathbf{D}^4}{8m^3}\}\Psi$$
$$-\frac{1}{4}c_1 F_{\mu\nu}F^{\mu\nu} + \frac{c_2}{m^2} g F_{\mu\nu} D^2 g F^{\mu\nu} + \frac{c_3}{m^2} g^3 f_{ABC} F^A_{\mu\nu} F^B_{\mu\alpha} F^C_{\nu\alpha}$$

$$\delta\mathcal{L}_{NRQCD} = \frac{d_{ss}}{m_1 m_2}\psi_1^\dagger\psi_1\chi_2^\dagger\chi_2 + \frac{d_{sv}}{m_1 m_2}\psi_1^\dagger\boldsymbol{\sigma}\psi_1\chi_2^\dagger\boldsymbol{\sigma}\chi_2$$
$$+\frac{d_{vs}}{m_1 m_2}\psi_1^\dagger \mathrm{T}^a\psi_1\chi_2^\dagger \mathrm{T}^a\chi_2 + \frac{d_{vv}}{m_1 m_2}\psi_1^\dagger \mathrm{T}^a\boldsymbol{\sigma}\psi_1\chi_2^\dagger \mathrm{T}^a\boldsymbol{\sigma}\chi_2\,.$$

Typically, $\quad c_i \sim 1 + \sum_n A_n \left(\alpha_s \ln\frac{m}{\nu}\right)^n \qquad d_i \sim \alpha_s\left(1 + \sum_n B_n\left(\alpha_s \ln\frac{m}{\nu}\right)^n\right)$

$\nu_p \gg |\mathbf{p}|$: quark-antiquark relative three-momentum.

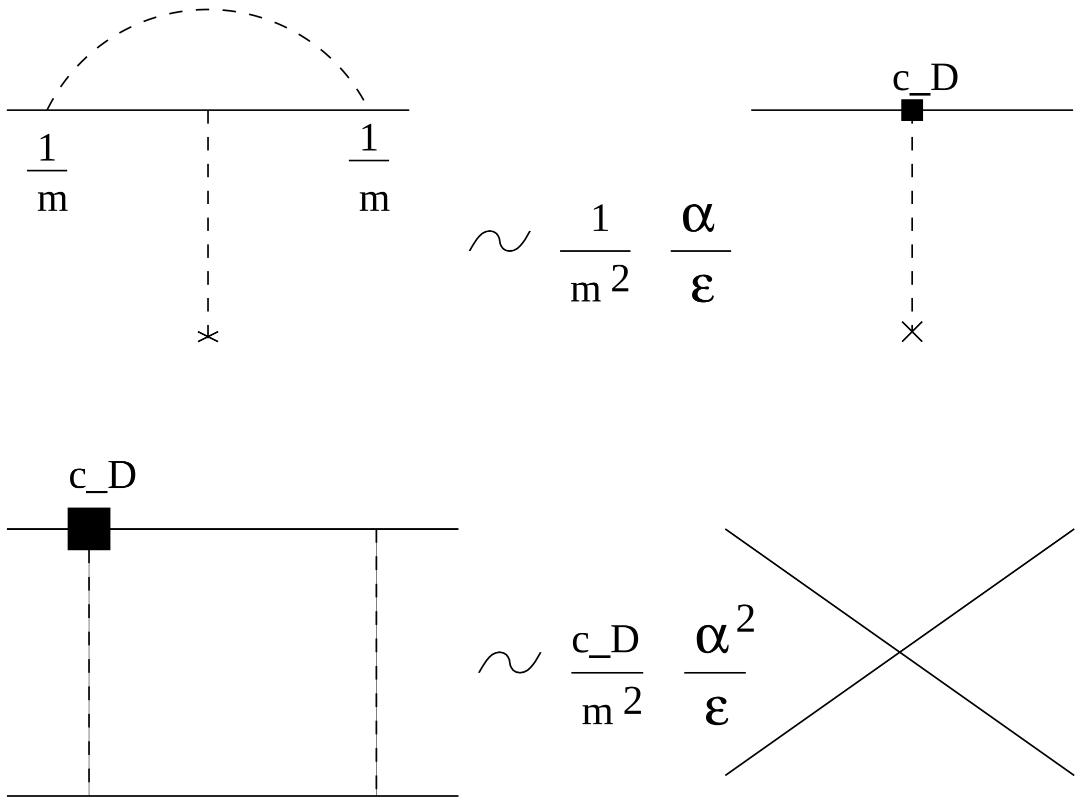
$\nu_s \gg |\mathbf{k}|$: gluon three-momentum, transfer momentum between the quark and antiquark.

$m \gg \nu_p \sim \nu_s$

Matching coefficients: $c(\nu_s)$, $d(\nu_s, \nu_p)$

LL $\rightarrow$ $c(\nu_s)$, $d(\nu_s)$

Running ν_s LL: HQET; $1/m$ expansion, $\dfrac{i}{q^0 + i\epsilon}$



$$\nu_s \frac{d}{d\nu_s} c_D = \frac{\alpha_s}{4\pi}\left[\frac{4C_A}{3}c_D - \left(\frac{2C_A}{3}+\frac{32C_f}{3}\right)c_k^2 - \frac{10C_A}{3}c_F^2 + \frac{8T_F n_f}{3}c_1^{hl}\right],$$

$$\begin{aligned}
\nu_s \frac{d}{d\nu_s} d_{ss} &= -2C_f\left(C_f - \frac{C_A}{2}\right)\alpha_s^2 c_k^2\,,\\
\nu_s \frac{d}{d\nu_s} d_{sv} &= 0\,,\\
\nu_s \frac{d}{d\nu_s} d_{vs} &= 4\left(C_f - C_A\right)\alpha_s^2 c_k^2 + \frac{3}{2}\alpha_s^2 C_A c_D\,,\\
\nu_s \frac{d}{d\nu_s} d_{vv} &= -\frac{C_A}{2}\alpha_s^2 c_F^2\,.
\end{aligned}$$

We define $z = \left[\frac{\alpha_s(\nu_s)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}} \simeq 1 - 1/(2\pi)\alpha_s(\nu_s)\ln(\frac{\nu_s}{m})$, $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$

$$\begin{aligned}
c_F(\nu_s) &= z^{-C_A}\,,\\
c_S(\nu_s) &= 2z^{-C_A} - 1\,,\\
c_D(\nu_s) &= \frac{9C_A}{9C_A + 8T_F n_f}\left\{-\frac{5C_A + 4T_F n_f}{4C_A + 4T_F n_f} z^{-2C_A} + \frac{C_A + 16C_f - 8T_F n_f}{2(C_A - 2T_F n_f)}\right.\\
&\quad + \frac{-7C_A^2 + 32C_A C_f - 4C_A T_F n_f + 32C_f T_F n_f}{4(C_A + T_F n_f)(2T_F n_f - C_A)} z^{4T_F n_f/3 - 2C_A/3}\\
&\quad \left. + \frac{8T_F n_f}{9C_A}\left[z^{-2C_A} + \left(\frac{20}{13} + \frac{32}{13}\frac{C_f}{C_A}\right)\left[1 - z^{\frac{-13C_A}{6}}\right]\right]\right\},
\end{aligned}$$

$$
\begin{aligned}
d_{ss}(\nu_s) &= d_{ss}(m) + 4C_f\left(C_f - \frac{C_A}{2}\right)\frac{\pi}{\beta_0}\alpha_s(m)\left[z^{\beta_0} - 1\right],\\
d_{sv}(\nu_s) &= d_{sv}(m),\\
d_{vs}(\nu_s) &= d_{vs}(m) - (C_f - C_A)\frac{8\pi}{\beta_0}\alpha_s(m)\left[z^{\beta_0} - 1\right]\\
&\quad - \frac{27C_A^2}{9C_A + 8T_F n_f}\frac{\pi}{\beta_0}\alpha_s(m)\left\{-\frac{5C_A + 4T_F n_f}{4C_A + 4T_F n_f}\frac{\beta_0}{\beta_0 - 2C_A}\left(z^{\beta_0 - 2C_A} - 1\right)\right.\\
&\qquad + \frac{C_A + 16C_f - 8T_F n_f}{2(C_A - 2T_F n_f)}\left(z^{\beta_0} - 1\right)\\
&\qquad + \frac{-7C_A^2 + 32C_A C_f - 4C_A T_F n_f + 32C_f T_F n_f}{4(C_A + T_F n_f)(2T_F n_f - C_A)}\\
&\qquad\quad \times \frac{3\beta_0}{3\beta_0 + 4T_F n_f - 2C_A}\left(z^{\beta_0 + 4T_F n_f/3 - 2C_A/3} - 1\right)\\
&\qquad + \frac{8T_F n_f}{9C_A}\left[\frac{\beta_0}{\beta_0 - 2C_A}\left(z^{\beta_0 - 2C_A} - 1\right) + \left(\frac{20}{13} + \frac{32}{13}\frac{C_f}{C_A}\right)\right.\\
&\qquad\quad \left.\left.\times\left(\left[z^{\beta_0} - 1\right] - \frac{6\beta_0}{6\beta_0 - 13C_A}\left[z^{\beta_0 - \frac{13C_A}{6}} - 1\right]\right)\right]\right\},\\
d_{vv}(\nu_s) &= d_{vv}(m) + \frac{C_A}{\beta_0 - 2C_A}\pi\alpha_s(m)\left\{z^{\beta_0 - 2C_A} - 1\right\}.
\end{aligned}
$$

Bauer-Manohar; Pineda

One equation for the soft running.

pNRQCD: the scale mv

The integration of the mv scale gives rise to potential terms. The Lagrangian is local in time but not in space.

Playing with the scales:

1) $mv \sim \Lambda_{QCD}$

2) $mv \gg \Lambda_{QCD} \gg mv^2$

3) $mv \gg mv^2 \sim \Lambda_{QCD}$

4) $mv \gg mv^2 \gg \Lambda_{QCD}$

Loosely speaking, when to trust the perturbative calculation and the size of NP corrections.

$mv \gg \Lambda_{QCD}$ ($\Upsilon(1S)$, t-$\bar{t}$, b-$\bar{b}$ sum rules)

- Degrees of freedom
- symmetries
- cutoff

pNRQCD has two ultraviolet cut-offs, ν_{us} and ν_p. ν_{us} fulfils the relation $\mathbf{p}^2/m \ll \nu_{us} \ll |\mathbf{p}|$ and is the cut-off of the energy of the quarks, and of the energy and the momentum of the gluons. ν_p fulfils $|\mathbf{p}| \ll \nu_p \ll m$ and is the cut-off of the relative momentum of the quark–antiquark system, $\mathbf{p}$.

Power counting/scales

Scales: m, p, $1/r$, $\Lambda_{mp} = \{\Lambda_{QCD}, mv^2, ...\}$

Dimensionless quantities:

$$\boxed{\frac{p}{m},\ \alpha_s,\ \frac{1}{m\,r},\ \boxed{\Lambda_{mp} r} \ll 1}$$

The multipole expansion can be used in the new EFT.

$$L_{pNRQCD} = L'_{NRQCD} + \int\int d^3x_1 d^3x_2 \psi(x_1)\chi_c(x_2) V(x_1 - x_2)\psi^\dagger(x_1)\chi_c^\dagger(x_2)$$

L'_{NRQCD}, gluons multipole expanded (only ultrasoft gluons).

$$V_s^{(0)} \equiv -C_F \frac{\alpha_{V_s}}{r}.$$

$$\frac{V_s^{(1)}}{m} \equiv -\frac{C_F C_A D_s^{(1)}}{2mr^2}.$$

$$\frac{V_s^{(2)}}{m^2} = -\frac{C_F D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r}, \mathbf{p}^2\right\} + \frac{C_F D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2 + \frac{\pi C_F D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r})$$
$$+\frac{4\pi C_F D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_F D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_F D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),$$

where $S_{12}(\hat{\mathbf{r}}) \equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\,\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ and $\mathbf{S} = \boldsymbol{\sigma}_1/2 + \boldsymbol{\sigma}_2/2$.

To go to the wave function description one has to project to the quark-antiquark sector.

$$\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi(\mathbf{x}_1,\mathbf{x}_2)\psi(x_1)\chi_c(x_2)|0\rangle$$

$$H\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi(\mathbf{x}_1,\mathbf{x}_2)\psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|0\rangle = \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 (\hat{h}\Psi(\mathbf{x}_1,\mathbf{x}_2))\psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|0\rangle$$

For QED

$$\begin{aligned} L_{pNRQED} &= \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi^\dagger(\mathbf{x}_1,\mathbf{x}_2)\left(iD_0 + \frac{\mathbf{D}^2_{\mathbf{x}_1}}{2m_1} + \frac{\mathbf{D}^2_{\mathbf{x}_2}}{2m_2} - V(\mathbf{x},\mathbf{p})\right)\Psi(\mathbf{x}_1,\mathbf{x}_2) \\ &= \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi^\dagger(\mathbf{x}_1,\mathbf{x}_2)\left(i\partial_0 + \frac{\boldsymbol{\nabla}^2_{\mathbf{x}}}{m} + \frac{\boldsymbol{\nabla}^2_{\mathbf{X}}}{4m}\right. \\ &\quad \left. - e\mathbf{x}\cdot\boldsymbol{\nabla}A_0(\mathbf{X}) - 2ie\frac{\mathbf{A}(\mathbf{X})\cdot\boldsymbol{\nabla}_{\mathbf{x}}}{m} - V(\mathbf{x},\mathbf{p})\right)\Psi(\mathbf{x}_1,\mathbf{x}_2) \end{aligned}$$

New fields: Singlet S, Octet (O) and US gluons.
Gauge transformation:

$$S(\mathbf{x},\mathbf{X},t) \to S(\mathbf{x},\mathbf{X},t)\,, \qquad O(\mathbf{x},\mathbf{X},t) \to g(\mathbf{X},t)O(\mathbf{x},\mathbf{X},t)g^{-1}(\mathbf{X},t)\,.$$

Field Redefinitions.

$$\Psi(\mathbf{x}_1,\mathbf{x}_2) = \phi(\mathbf{x}_1,\mathbf{x}_2)S(\mathbf{x},\mathbf{X}) + \phi(\mathbf{x}_1,\mathbf{X})O(\mathbf{x},\mathbf{X})\phi(\mathbf{X},\mathbf{x}_2)$$

$$\phi(\mathbf{y},\mathbf{x},t) \equiv \mathrm{P}\exp\left\{ig\int_0^1 ds\,(\mathbf{y}-\mathbf{x})\cdot\mathbf{A}(\mathbf{x}-s(\mathbf{x}-\mathbf{y}),t)\right\}$$

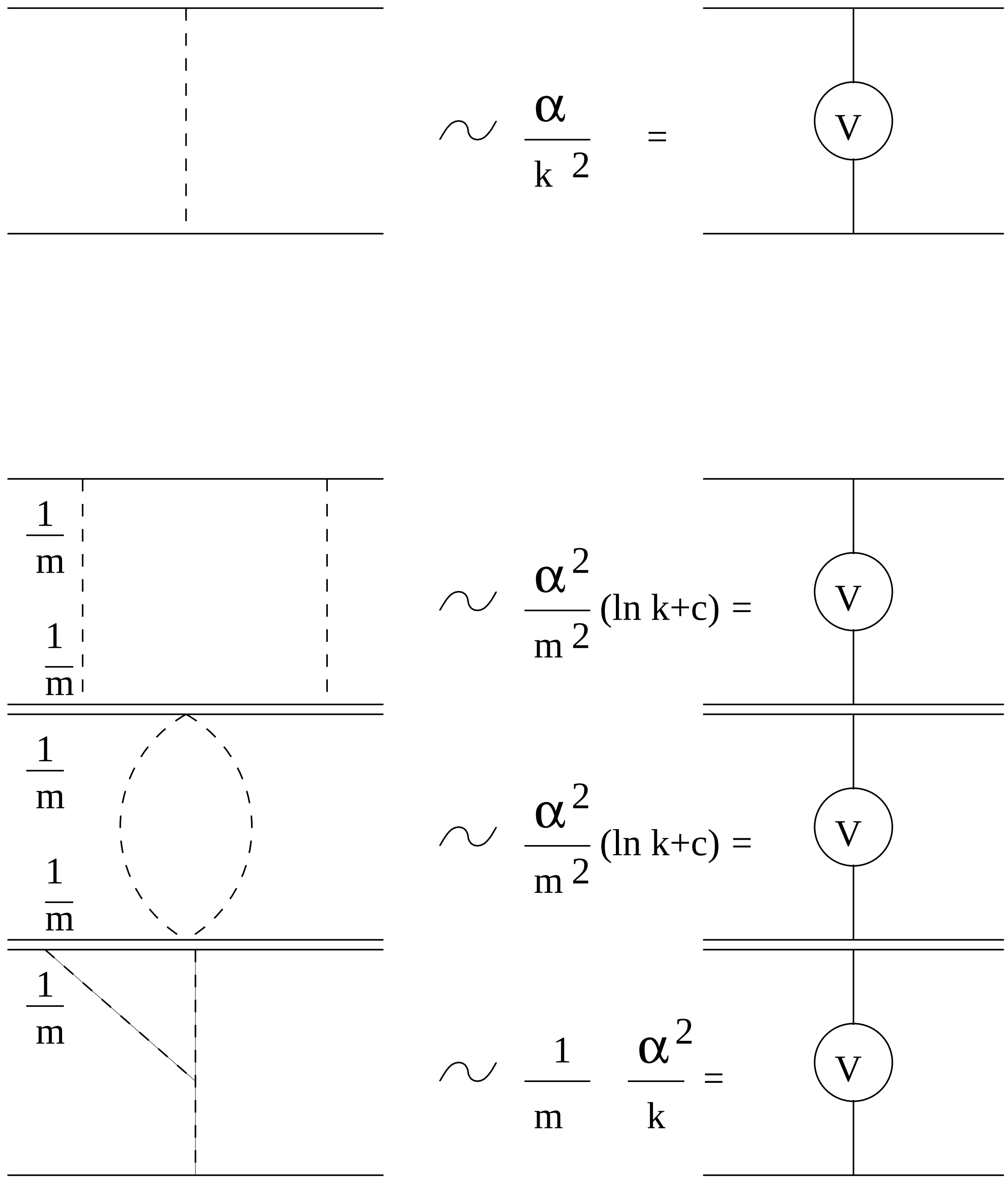
pNRQCD Lagrangian at $O(r)$

$$
\begin{aligned}
\mathcal{L}_{pNRQCD} &= \mathrm{Tr}\{S^\dagger \left(i\partial_0 - V_s^{(0)}(\mathbf{x})\right) S + O^\dagger \left(iD_0 - V_o^{(0)}(\mathbf{x})\right) O\} \\
&+ gV_A(\mathbf{x})\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, S + S^\dagger \mathbf{x}\cdot\mathbf{E}\, O\right\} + g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, O + O^\dagger O \mathbf{x}\cdot\mathbf{E}\right\} \\
&- \mathrm{Tr}\{S^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}(\mathbf{x})}{m^n}\right) S - O^\dagger \left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_o^{(n)}(\mathbf{x})}{m^n}\right) O\},
\end{aligned}
$$

Interpolating fields:

$$
Q_2^\dagger(\mathbf{x}_2, t)\phi(\mathbf{x}_2, \mathbf{x}_1; t)Q_1(\mathbf{x}_1, t) = Z_s^{1/2}(\mathbf{x})S(\mathbf{X}, \mathbf{x}, t)
$$

$$
Q_2^\dagger(x_2)\phi(\mathbf{x}_2, \mathbf{X}; t)T^a\phi(\mathbf{X}, \mathbf{x}_1; t)Q_1(x_1) = Z_o^{1/2}(\mathbf{x})O^a(\mathbf{X}, \mathbf{x}, t)
$$



$\sim \frac{\alpha}{k^2} =$
V
$\frac{1}{m}$
$\frac{1}{m}$
$\sim \frac{\alpha^2}{m^2}(\ln k + c) =$
V
$\frac{1}{m}$
$\frac{1}{m}$
$\sim \frac{\alpha^2}{m^2}(\ln k + c) =$
V
$\frac{1}{m}$
$\sim \frac{1}{m} \frac{\alpha^2}{k} =$
V

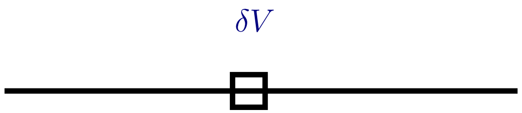
Renormalization group in pNRQCD (LL) (Ultrasoft running)

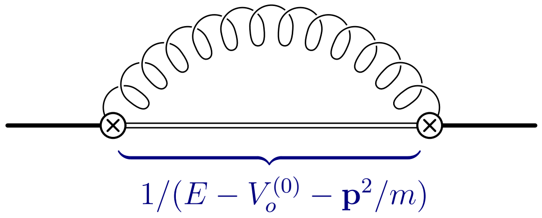
Aim: to obtain the running of the pNRQCD matching coefficients: $(\alpha_s \ln)^n$, $\alpha_s (\alpha_s \ln)^n$

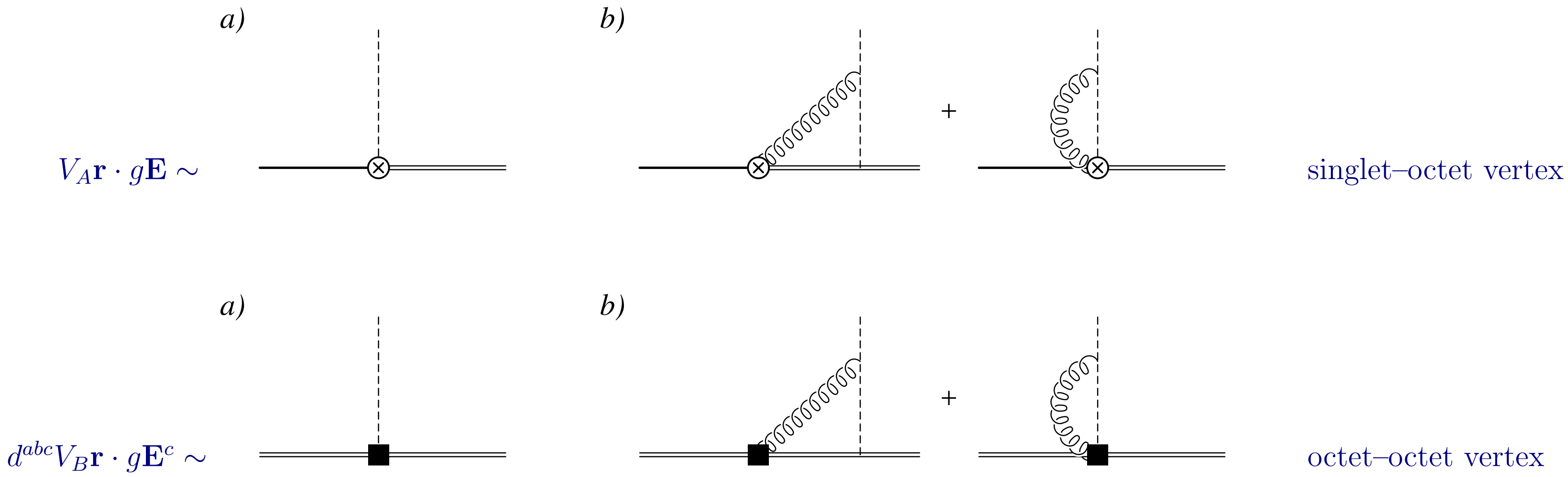
Relevant for:

- Spectrum: heavy quarkonium and QED.
- Currents: electromagnetic decays.
- Currents: Normalization of bottomonium sum rules.
- Currents: Normalization of t-$\bar{t}$ production near threshold.

$\nu_p \gg |\mathbf{p}|$: quark-antiquark relative three-momentum.
$\nu_{us} \gg |\mathbf{k}|$: gluon three-momentum.
$|\mathbf{p}| \gg \nu_{us} \gg \mathbf{p}^2/m$
Matching coefficients: $\tilde{V}(d(\nu_p,\nu_s,m),c(\nu_s,m),\nu_s,\nu_{us},r) = \tilde{V}(\nu_p,m,\nu_{us},r) \equiv \tilde{V}(\nu_p,\nu_{us})\,.$
$\nu_s \frac{d}{d\nu_s}\tilde{V} = 0$; $\nu_s = 1/r$
LL: $\nu_p \frac{d}{d\nu_p}\tilde{V} = 0$
ν_{us}. The computation can be formally organized through the multipole expansion.







Corrections to the Green Function

$$G_s(E) = P_s \frac{1}{H - H_I - E} P_s = G_s^{(0)} + \delta G_s$$

From the potential:

$$\delta G_s \sim \frac{1}{H_s - E} \delta V \frac{1}{H_s - E}$$

From ultrasoft gluons:

$$\begin{aligned}\delta G_s \ &\sim \ \frac{1}{H_s - E} \int \frac{d^3\mathbf{k}}{(2\pi)^{D-1}} \mathbf{r} \frac{k}{k + H_o - E} \mathbf{r} \frac{1}{H_s - E} \\ &\sim \ \frac{1}{H_s - E} \mathbf{r} (H_o - E)^3 \left\{ \frac{1}{\epsilon} + \gamma + \ln \frac{(H_o - E)^2}{\nu^2} + C \right\} \mathbf{r} \frac{1}{H_s - E}\end{aligned}$$

$$
\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_s} &= \frac{2}{3}\frac{\alpha_s}{\pi}V_A^2\left(\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right)^3, \\
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_o} &= \frac{2}{3}\frac{\alpha_s}{\pi}V_A^2\left(\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right)^3, \\
\nu_{us}\frac{d}{d\nu_{us}}\alpha_s &= -\beta_0\frac{\alpha_s^2}{2\pi}, \\
\nu_{us}\frac{d}{d\nu_{us}}V_A &= 0, \\
\nu_{us}\frac{d}{d\nu_{us}}V_B &= 0\,.
\end{aligned}
$$

$$
\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}C_A D_s^{(1)} &= \frac{16}{3}\frac{\alpha_s}{\pi}V_A^2 c_k\left[\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right]\left[2C_f\alpha_{V_s}+\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}\right], \\
\nu_{us}\frac{d}{d\nu_{us}}D_{d,s}^{(2)} &= \frac{16}{3}\frac{\alpha_s}{\pi}V_A^2 c_k^2\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}, \\
\nu_{us}\frac{d}{d\nu_{us}}D_{1,s}^{(2)} &= \frac{8}{3}\frac{\alpha_s}{\pi}V_A^2 c_k^2\left[\left(\frac{C_A}{2}-C_f\right)\alpha_{V_o}+C_f\alpha_{V_s}\right],
\end{aligned}
$$

and zero for the other matching coefficients (in particular for the spin-dependent potentials).
Soto-Pineda; Pineda

RG equations within an strict expansion in α

$$\begin{aligned}
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_s} &= \frac{2}{3}\frac{\alpha_s(\nu_{us})}{\pi}\left(\frac{C_A}{2}\right)^3\alpha_s^3(r^{-1})\,,\\
\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_o} &= 0\,,\\
\nu_{us}\frac{d}{d\nu_{us}}C_A D_s^{(1)} &= \frac{16}{3}\frac{\alpha_s(\nu_{us})}{\pi}\frac{C_A}{2}\left(C_f+\frac{C_A}{2}\right)\alpha_s^2(r^{-1})\,,\\
\nu_{us}\frac{d}{d\nu_{us}}D_{1,s}^{(2)} &= \frac{8}{3}\frac{\alpha_s(\nu_{us})}{\pi}\frac{C_A}{2}\alpha_s(r^{-1})\,,\\
\nu_{us}\frac{d}{d\nu_{us}}D_{d,s}^{(2)} &= \frac{16}{3}\frac{\alpha_s(\nu_{us})}{\pi}\left(\frac{C_A}{2}-C_f\right)\alpha_s(r^{-1})\,.
\end{aligned}$$

Initial conditions ($\nu_{us} = 1/r$):

$$\begin{aligned}
\alpha_{V_s}(r^{-1}) &= \alpha_s(r^{-1}) \left\{ 1 + (a_1 + 2\gamma_E\beta_0) \frac{\alpha_s(r^{-1})}{4\pi} \right. \\
&\left. + \left[\gamma_E \left(4a_1\beta_0 + 2\beta_1\right) + \left(\frac{\pi^2}{3} + 4\gamma_E^2\right)\beta_0^2 + a_2 \right] \frac{\alpha_s^2(r^{-1})}{16\,\pi^2} \right\}, \\
D_s^{(1)}(r^{-1}) &= \alpha_s^2(r^{-1}), \\
D_{1,s}^{(2)}(r^{-1}) &= \alpha_s(r^{-1}), \\
D_{2,s}^{(2)}(r^{-1}) &= \alpha_s(r^{-1}), \\
D_{d,s}^{(2)}(r^{-1}) &= \alpha_s(r^{-1})(2 + c_D(r^{-1}) - 2c_F^2(r^{-1})) \\
&+ \frac{1}{\pi}\left[d_{vs}(r^{-1}) + 3d_{vv}(r^{-1}) + \frac{1}{C_f}(d_{ss}(r^{-1}) + 3d_{sv}(r^{-1})) \right], \\
D_{S^2,s}^{(2)}(r^{-1}) &= \alpha_s(r^{-1})c_F^2(r^{-1}) - \frac{3}{2\pi C_f}(d_{sv}(r^{-1}) + C_f d_{vv}(r^{-1})), \\
D_{LS,s}^{(2)}(r^{-1}) &= \frac{\alpha_s(r^{-1})}{3}(c_S(r^{-1}) + 2c_F(r^{-1})), \\
D_{S_{12},s}^{(2)}(r^{-1}) &= \alpha_s(r^{-1})c_F^2(r^{-1}), \\
\alpha_{V_o}(r^{-1}) &= \alpha_s(r^{-1}), \\
V_A(r^{-1}) &= 1,
\end{aligned}$$

The RG improved potentials for the singlet read:

$$
\begin{aligned}
\alpha_{V_s}(\nu_{us}) &= \alpha_{V_s}(r^{-1}) + \frac{C_A^3}{6\beta_0}\alpha_s^3(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_s^{(1)}(\nu_{us}) &= D_s^{(1)}(r^{-1}) + \frac{16}{3\beta_0}\left(\frac{C_A}{2}+C_f\right)\alpha_{\rm s}^2(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_{1,s}^{(2)}(\nu_{us}) &= D_{1,s}^{(2)}(r^{-1}) + \frac{8C_A}{3\beta_0}\alpha_{\rm s}(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_{2,s}^{(2)}(\nu_{us}) &= D_{2,s}^{(2)}(r^{-1}), \\
D_{d,s}^{(2)}(\nu_{us}) &= D_{d,s}^{(2)}(r^{-1}) + \frac{32}{3\beta_0}\left(\frac{C_A}{2}-C_f\right)\alpha_s(r^{-1})\log\left(\frac{\alpha_s(r^{-1})}{\alpha_s(\nu_{us})}\right), \\
D_{S^2,s}^{(2)}(\nu_{us}) &= D_{S^2,s}^{(2)}(r^{-1}), \\
D_{LS,s}^{(2)}(\nu_{us}) &= D_{LS,s}^{(2)}(r^{-1}), \\
D_{S_{12},s}^{(2)}(\nu_{us}) &= D_{S_{12},s}^{(2)}(r^{-1}).
\end{aligned}
$$

Soto, Pineda; Pineda

OBSERVABLE: NNLL heavy quarkonium mass $O(m\alpha^{4+n}\ln^n\alpha)$ (Pineda; Hoang-Stewart)

$$\delta E^{\rm pot}_{n,l,j}(\nu_{us}) = E_n\alpha_s^2\left\{-\frac{2C_A}{3\beta_0}\left[\frac{C_A^2}{2}+4C_AC_f\frac{1}{n(2l+1)}+2C_f^2\left(\frac{8}{n(2l+1)}-\frac{1}{n^2}\right)\right]\log\left(\frac{\alpha_s(\nu_{us})}{\alpha_s}\right)\right.$$
$$+\frac{C_f^2\delta_{l0}}{3n}\left(-\frac{16}{\beta_0}\left[C_f-\frac{C_A}{2}\right]\log\left(\frac{\alpha_s(\nu_{us})}{\alpha_s}\right)\right.$$
$$\left.-\frac{3}{2}(1+c_D-2c_F^2)-\frac{3}{2\pi\alpha_s}\left[d_{vs}+3d_{vv}+\frac{1}{C_f}(d_{ss}+3d_{sv})\right]\right)$$
$$-\frac{4}{3}\frac{C_f^2\delta_{l0}\delta_{s1}}{n}\left\{z^{-2C_A}-1+\frac{3}{2}\frac{C_A}{\beta_0-2C_A}\left[z^{-\beta_0}-z^{-2C_A}\right]\right\}$$
$$\left.-\frac{(1-\delta_{l0})\delta_{s1}}{l(2l+1)(l+1)n}C_{j,l}\frac{C_f^2}{2}\right\},$$

where $E_n=-mC_f^2\alpha_s^2/(4n^2)$, $\nu_s=2a_n^{-1}$ where $2a_n^{-1}=\frac{mC_f\alpha_s(2a_n^{-1})}{n}$, and

$$C_{j,l}=\begin{cases}-\frac{(l+1)}{2\,l-1}\left\{4(2l-1)\left(z^{-C_A}-1\right)+\left(z^{-2C_A}-1\right)\right\} & ,\ j=l-1\\ -4\left(z^{-C_A}-1\right)+\left(z^{-2C_A}-1\right) & ,\ j=l\\ \frac{l}{2\,l+3}\left\{4(2l+3)\left(z^{-C_A}-1\right)-\left(z^{-2C_A}-1\right)\right\} & ,\ j=l+1.\end{cases}$$

Check with $O(m\alpha^5\ln\alpha)$ known logs: Brambilla, Vairo, Soto, Pineda; Kniehl, Penin; Hoang, Manohar and Stewart.

Muonic Hydrogen mass at NNLL. Check with known logs by Pachucki.

One equation for the ultrasoft running.

Renormalization group in pNRQCD (NLL) (potential running)

ν_p enters into the game.
The running on ν_p can be obtained from pNRQCD (there is no running of ν_p from NRQCD to pNRQCD). It can be obtained by Quantum mechanics computations. Example: iteration of the potentials. Divergent integrals in $|\mathbf{p}|$ and r.

$$h_s = c_k\frac{\mathbf{p}^2}{m} - C_f\frac{\alpha_{V_s}}{r} - c_4\frac{\mathbf{p}^4}{4m^3} - \frac{C_f C_A D_s^{(1)}}{2mr^2} - \frac{C_f D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r}, \mathbf{p}^2\right\} + \frac{C_f D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2$$
$$+\frac{\pi C_f D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r}) + \frac{4\pi C_f D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_f D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_f D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),$$

where $C_f = (N_c^2-1)/(2N_c)$ and $c_k = c_4 = 1$ (we only use c_4 for tracking of the contribution due to this term). The propagator of the singlet is (formally)

$$\frac{1}{E - h_s}.$$

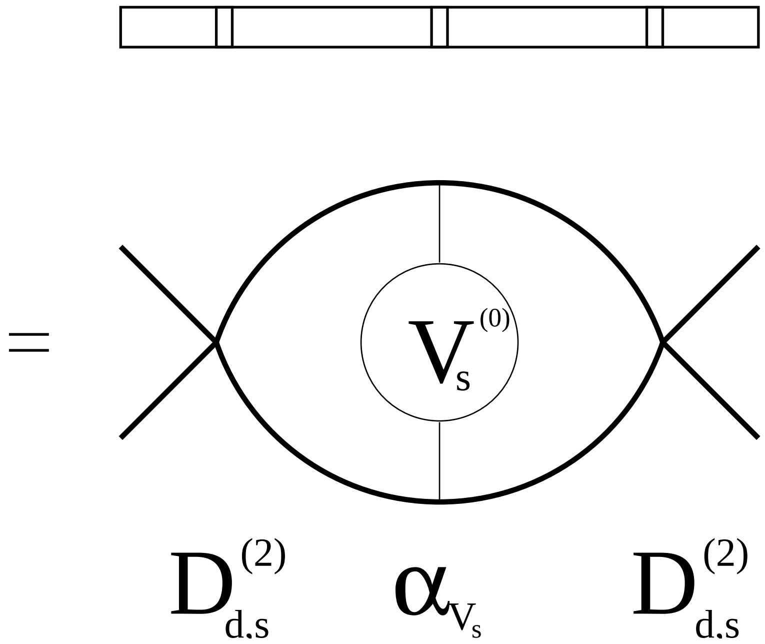
At leading order (within an strict expansion in α_s) the propagator of the singlet reads

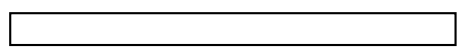
$$= G_c(E) = \frac{1}{E - h_s^{(0)}} = \frac{1}{E - \mathbf{p}^2/m - C_f\alpha_s/r}.$$

If we were interested in computing the spectrum at $O(m\alpha_s^6)$, one should consider the iteration of subleading potentials (δh_s) in the propagator as follows:

$$G_c(E)\delta h_s G_c(E)\cdots\delta h_s G_c(E).$$

In general, these contributions will produce logarithmic divergences due to potential loops. These divergences can be absorbed in the matching coefficients, $D_{d,s}^{(2)}$ and $D_{S^2,s}^{(2)}$, of the local potentials (proportional to the $\delta^{(3)}(\mathbf{r})$) providing with the renormalization group equations of these matching coefficients in terms of ν_p. Let us explain how it works in detail. Since the singular behavior of the potential loops appears for $\mathbf{p}^2/m \gg \alpha_s/r$, a perturbative expansion in α_s is licit in $G_c(E)$, which can be approximated by



 $= G_c^{(0)}(E) = \frac{1}{E - \mathbf{p}^2/m}$.

$$\langle \mathbf{r}=0| \frac{1}{E-\mathbf{p}^2/m} C_f \frac{\alpha_{V_s}}{r} \frac{1}{E-\mathbf{p}^2/m} |\mathbf{r}=0\rangle$$
$$\sim \int \frac{\mathrm{d}^d p'}{(2\pi)^d} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{m}{\mathbf{p}'^2 - mE} C_f \frac{4\pi\alpha_{V_s}}{\mathbf{q}^2} \frac{m}{\mathbf{p}^2 - mE} \sim -C_f \frac{m^2 \alpha_{V_s}}{16\pi} \frac{1}{\epsilon},$$

where $D = 4 + 2\epsilon$ and $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. This divergence is absorbed in $D_{d,s}^{(2)}$ contributing to its running at NLL order as follows

$$\nu_p \frac{d}{d\nu_p} D^{(2)}_{d,s}(\nu_p) \sim \alpha_{V_s}(\nu_p) D^{(2)2}_{d,s}(\nu_p) + \cdots .$$

$O(m\alpha^8 \ln^3 \alpha)$ correction to the Hydrogen atom spectrum. Manohar-Stewart; Pineda

$|\mathbf{p}| \gg \nu_{us} \gg \mathbf{p}^2/m \to \nu_{us} = {\nu_p}^2/m$ (Luke, Manohar, Rothstein)

We can not lower ν_{us} further. Fight between two terms.

$$\frac{1}{\mathbf{p}^2/m + k}$$

$$\begin{aligned} \tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_p^2/m, r) \;\simeq\; & \tilde{V}(c(\nu_p), d(\nu_p, \nu_p), \nu_p, \nu_p^2/m, \nu_p) \\ & + \ln(\nu_p r) r \frac{d}{dr} \tilde{V}|_{1/r=\nu_p} + \cdots . \end{aligned}$$

and one equation for the potential running

One equation for the soft running,
one equation for the ultrasoft running,
and one equation for the potential running,
which rules them all and at the hard scale binds them.

Nonrelativistic Sum rules (b-$\bar{b}$, c-$\bar{c}$), t-$\bar{t}$ production near threshold

$$J^\mu = \bar{Q}\gamma^\mu Q = B_1 \psi^\dagger \boldsymbol{\sigma} \chi + \cdots$$

$$B_1 = 1 + a_1\alpha_s + a_2\alpha_s^2 + \cdots$$

Hoang(QED); Beneke, Signer, Smirnov; Czarnecki, Melnikov

$$\Gamma(V \to e^+e^-) \sim \frac{1}{m^2} B_1^2 |\phi(\mathbf{0})|^2$$

$$(q_\mu q_\nu - g_{\mu\nu})\Pi(q^2) = i \int d^4x e^{iqx} \langle \text{vac}|J_\mu(x) J_\nu(0)|\text{vac}\rangle$$

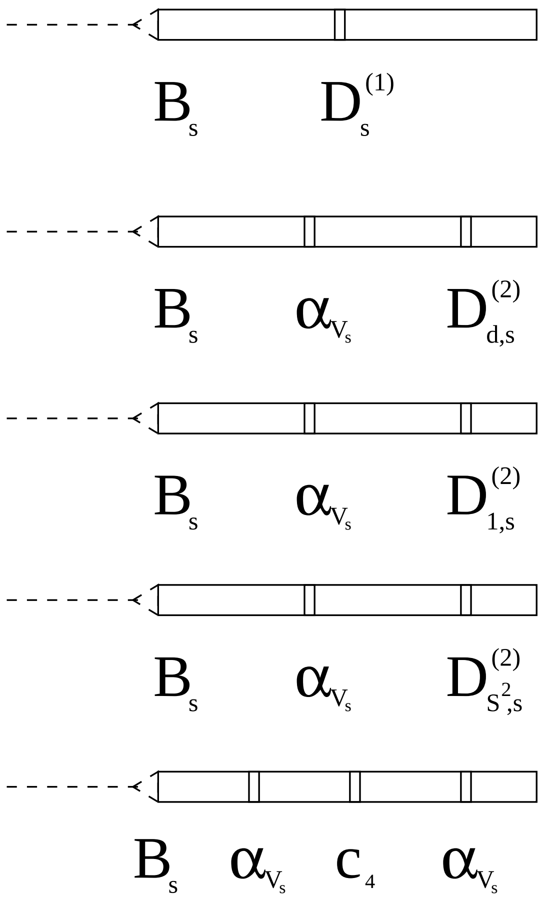
$$\Pi(q^2) \sim B_1^2 \langle \mathbf{r} = \mathbf{0}| \frac{1}{E - H} |\mathbf{r} = \mathbf{0}\rangle$$

Relation of $\sigma_{t\bar{t}}$ with $\Gamma_{NLL}(V_Q(nS) \to e^+e^-)$ and $M_{NNLL}(V_Q(nS))$

$$\sigma_{t-\bar{t}} \sim B_1(\nu)^2 \text{Im} G(0, 0, \sqrt{s}) + \cdots$$

$$G(0,0,E) = \sum_{m=0}^{\infty} \frac{|\phi_{0m}(0)|^2}{E_{0m} - E + i\epsilon - i\Gamma_t} + \frac{1}{\pi}\int_0^\infty dE' \frac{|\phi_{0E'}(0)|^2}{E_{0E'} - E + i\epsilon - i\Gamma_t}$$

Matching coefficient of the electromagnetic current at NLL



$$\nu_p \frac{d}{d\nu_p} B_s = -\frac{C_A C_f}{2} D_s^{(1)} - \frac{C_f^2}{4} \alpha_s \left\{ \alpha_s - \frac{4}{3} s(s+1) D^{(2)}_{S^2,s} - D^{(2)}_{d,s} + 4 D^{(2)}_{1,s} \right\} ,$$

$$b_1(m) = 1 - 2C_f \frac{\alpha_s(m)}{\pi}, \qquad b_0(m) = 1 + \left(\frac{\pi^2}{4} - 5 \right) \frac{C_f}{2} \frac{\alpha_s(m)}{\pi}.$$

The solution reads (Pineda; Hoang-Stewart)

$$B_s(\nu_p) = b_s(m) + A_1\frac{\alpha_s(m)}{w^{\beta_0}}\ln(w^{\beta_0}) + A_2\alpha_s(m)[z^{\beta_0}-1] + A_3\alpha_s(m)[z^{\beta_0-2C_A}-1] + A_4\alpha_s(m)[z^{\beta_0-13C_A/6}-1] + A_5\alpha_s(m)\ln(z^{\beta_0})\,,$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$, $z = \left[\frac{\alpha_s(\nu_p)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}}$ and $w = \left[\frac{\alpha_s(\nu_p^2/m)}{\alpha_s(\nu_p)}\right]^{\frac{1}{\beta_0}}$. The coefficients A_i read

$$A_1 = \frac{8\pi C_f}{3\beta_0^2}\left(C_A^2 + 2C_f^2 + 3C_f C_A\right)\,,$$

$$A_2 = \frac{\pi C_f[3\beta_0(26C_A^2 + 19C_A C_f - 32C_f^2) - C_A(208C_A^2 + 651C_A C_f + 116C_f^2)]}{78\,\beta_0^2\,C_A}\,,$$

$$A_3 = -\frac{\pi C_f^2[\beta_0(4s(s+1)-3) + C_A(15-14s(s+1))]}{6(\beta_0 - 2C_A)^2}\,,$$

$$A_4 = \frac{24\pi C_f^2(3\beta_0 - 11C_A)(5C_A + 8C_f)}{13\,C_A(6\beta_0 - 13C_A)^2}\,,$$

$$A_5 = \frac{-\pi C_f^2}{\beta_0^2\,(6\beta_0 - 13C_A)(\beta_0 - 2C_A)}\{C_A^2(-9C_A + 100C_f) + \beta_0 C_A(-74C_f + C_A(42 - 13s(s+1))) + 6\beta_0^2(2C_f + C_A(-3 + s(s+1)))\}\,.$$

Leading (Czarnecki-Melnikov; Beneke-Signer-Smirnov) and subleading (Kniehl-Penin) logs correct.

Inclusive decays to leptons and photons at NLL (Pineda)

By setting $\nu_p \sim m\alpha_s$, $B_s(\nu_p)$ includes all the large logs at NLL order in any (inclusive enough) S-wave heavy-quarkonium production observable we can think of. For instance, the decays to e^+e^- and to two photons at NLL $O(\alpha^{1+n}\ln^n\alpha)$ order read

$$\begin{aligned}
\Gamma(V_Q(nS)\to e^+e^-) &= 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{B_1(\nu_p)(1+\delta\phi_n)\}^2 \\
&\simeq 2\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{1+2(B_1(\nu_p)-1)+2\delta\phi_n\}\ , \\
\Gamma(P_Q(nS)\to\gamma\gamma) &= 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{B_0(\nu_p)(1+\delta\phi_n)\}^2 \\
&\simeq 6\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{1+2(B_0(\nu_p)-1)+2\delta\phi_n\}\ ,
\end{aligned}$$

where V and P stand for the vector and pseudoscalar heavy quarkonium, we have fixed $\nu_p = m_Q C_f\alpha_s/n$, $\alpha_s = \alpha_s(\nu_p)$, and ($\Psi_n(z) = \frac{d^n \ln\Gamma(z)}{dz^n}$ and $\Gamma(z)$ is the Euler Γ-function)

$$\delta\phi_n = \frac{\alpha_s}{\pi}\left[-C_A + \frac{\beta_0}{4}\left(\Psi_1(n+1) - 2n\Psi_2(n) + \frac{3}{2} + \gamma_E + \frac{2}{n}\right)\right].$$

NLL Hyperfine splitting

Kniehl, Penin, Smirnov, Steinhauser, Pineda;
Penin, Smirnov, Steinhauser, Pineda

$$\delta E \sim m\alpha^4 + m\alpha^5 \ln\alpha + m\alpha^6 \ln^2\alpha + \cdots$$
$$+ m\alpha^5 + m\alpha^6 \ln\alpha + m\alpha^7 \ln^2\alpha + m\alpha^8 \ln^3\alpha + \cdots$$

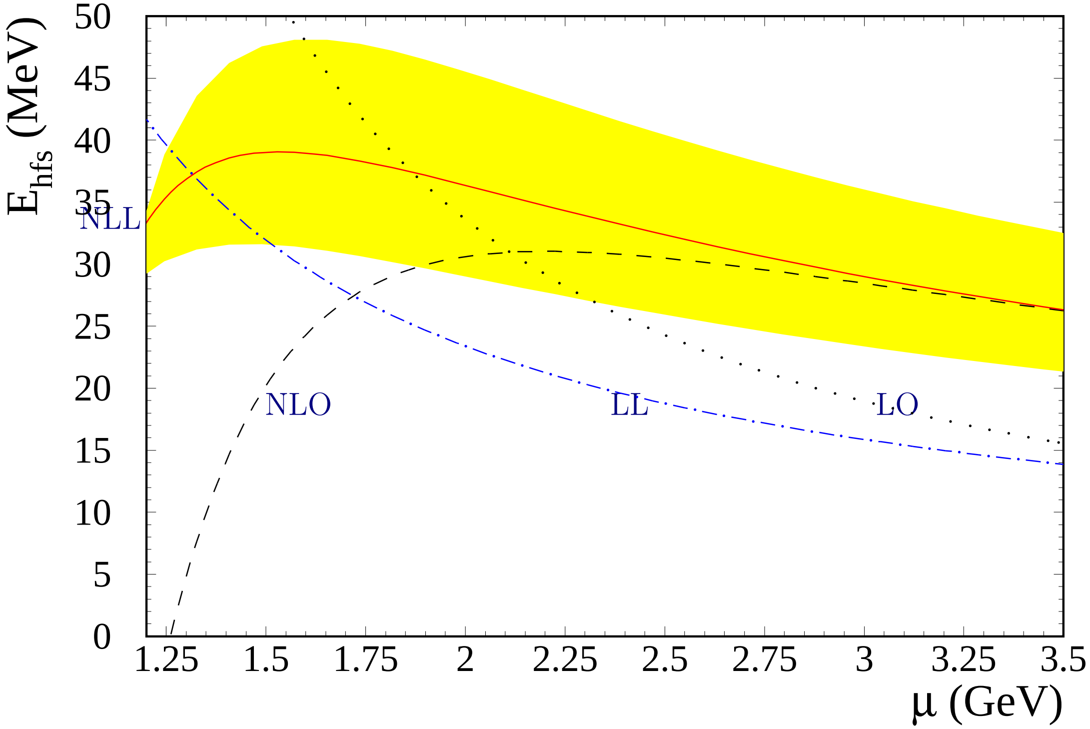


Figure 1: HFS for 1S bottomonium as the function of the renormalization scale μ in LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximation. For the NLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$.

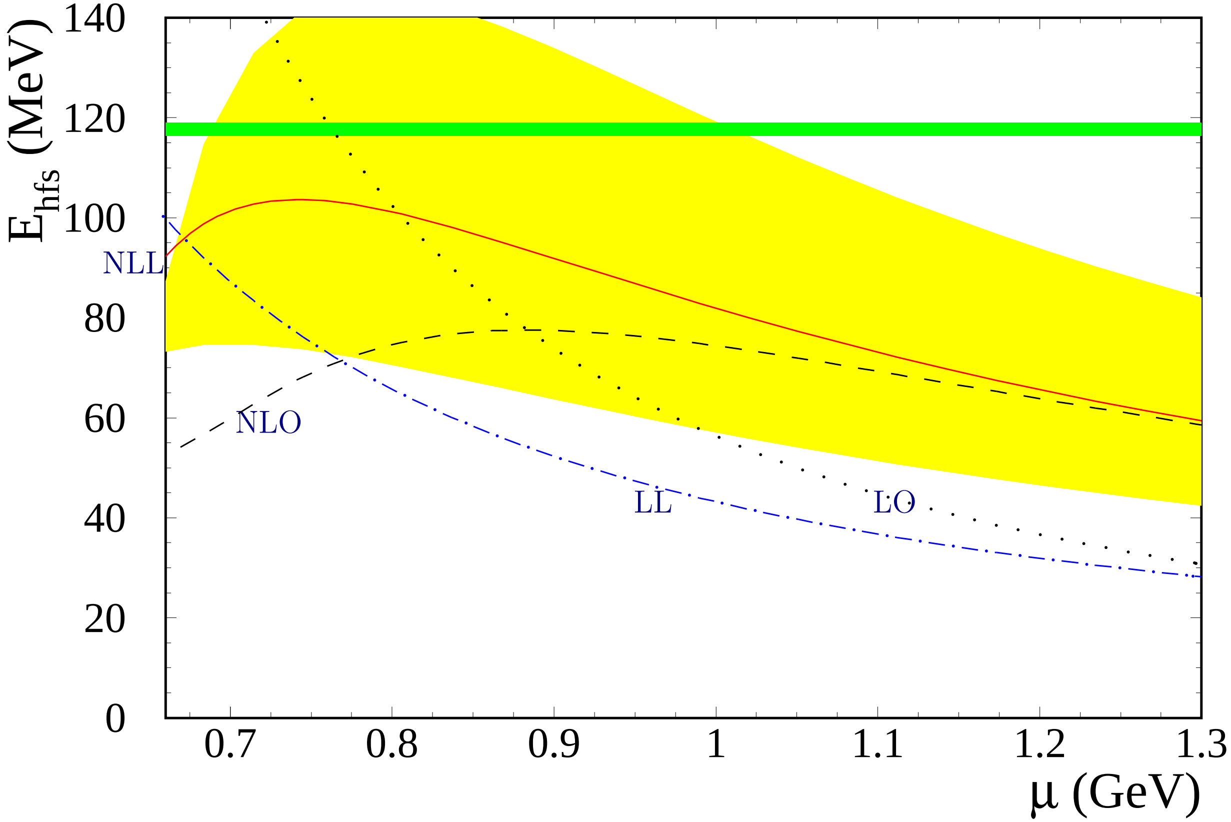


Figure 2: HFS for 1S charmonium as the function of the renormalization scale μ in LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximation versus the experimental result. For the NLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$. The horizontal band gives the experimental value and its error.

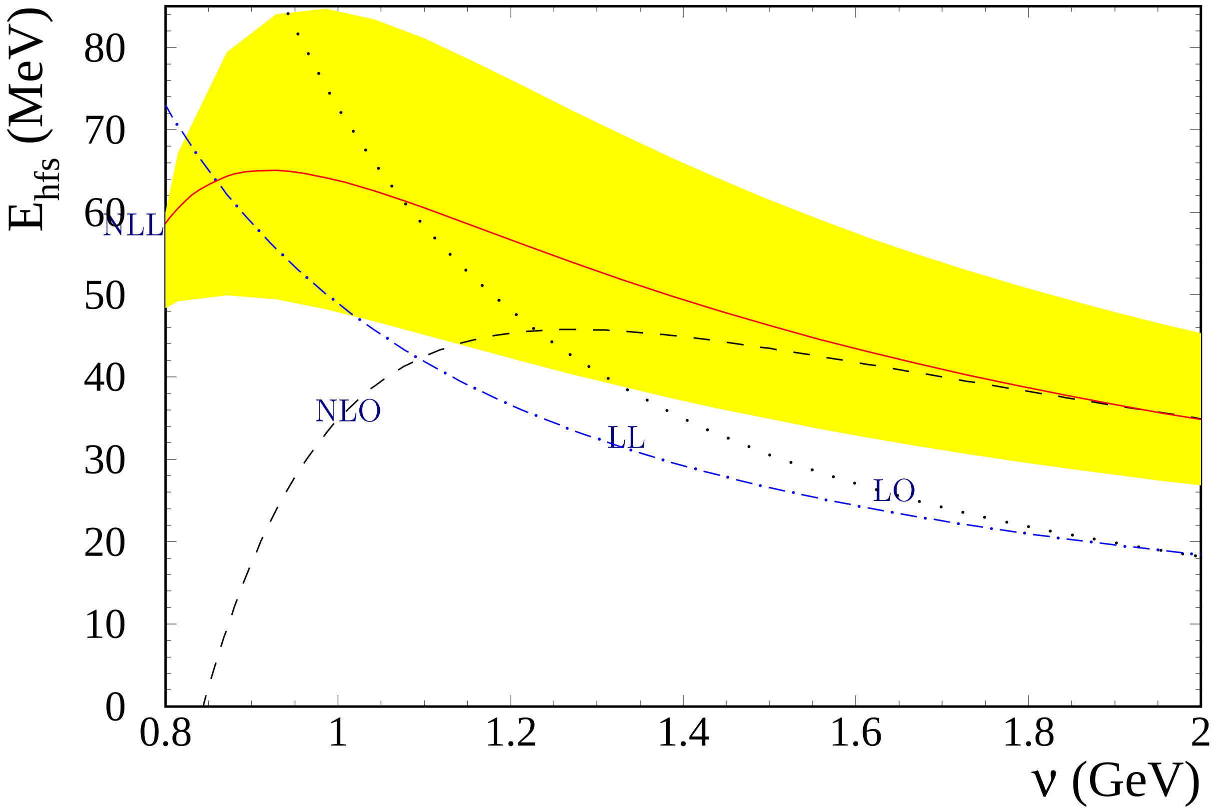


Figure 3: HFS for charm-bottom quarkonium as the function of the renormalization scale ν in LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximation for $\nu_h = 1.95$ GeV. For the NLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$.

$$M(\eta_b) = 9421 \pm 11\,(\mathrm{th})\,{}^{+9}_{-8}\,(\delta\alpha_s)\ \mathrm{MeV}$$

$$M(B_c^*) - M(B_c) = 65 \pm 24\,(\mathrm{th})\,{}^{+19}_{-16}\,(\delta\alpha_s)\ \mathrm{MeV}$$

NLL Hyperfine splitting; analytic result

$$E_{HF} = -\frac{4}{3}E_n\alpha_s\frac{C_f^2\delta_{l0}\delta_{s1}}{n}\Bigg\{(1+2\delta\phi_n)\alpha_s(m)\left(1+\frac{2\beta_0-7C_A}{2\beta_0-4C_A}\left\{z^{-2C_A+\beta_0}-1\right\}\right)$$
$$+\left(D^{(2)}_{S^2,s}(\nu_s=\nu_p=m)-\alpha_s(m)\right)$$
$$+\left(\ln\left(\frac{mC_f\alpha_s}{\nu_p n}\right)+\sum_{k=1}^{n}\frac{1}{k}+\frac{n-1}{2n}\right)\alpha_s c_F^2\gamma_{D^{(2)}_{S^2,s}}+\delta D^{(2)}_{S^2,s}(\nu_s)|_{NLL}+\delta D^{(2)}_{S^2,s}(\nu_p)|_{NLL}\Bigg\}\,,$$

where $\Psi_n(z)=\frac{d^n\ln\Gamma(z)}{dz^n}$ and $\Gamma(z)$ is the Euler Γ-function and

$$\delta\phi_n=\frac{\alpha_s}{\pi}\left[-C_A+\frac{\beta_0}{4}\left(3\ln\left(\frac{\nu_p n}{mC_f\alpha_s}\right)+\Psi_1(n+1)-2n\Psi_2(n)+\frac{3}{2}+\gamma_E+\frac{2}{n}\right)\right].$$

$$D^{(2)}_{S^2,s}(\nu_s=\nu_p=m)=\alpha_s(m)\left\{1+\left[-\frac{5}{9}T_Fn_f+\frac{3}{2}(1-\ln 2)T_F+\frac{11C_A-9C_f}{18}\right]\frac{\alpha_s(m)}{\pi}\right\}$$

$$\delta D^{(2)}_{S^2,s}(\nu_s)|_{NLL}=B_1\alpha_s^2(m)(z^{-\gamma_0+\beta_0}-1)+B_2\alpha_s^2(m)(z^{-\gamma_0+2\beta_0}-1)$$

where

$$B_1 = \frac{\beta_1\,\gamma_0-\beta_0\,(2\,\beta_0\,c_1+\gamma_1)}{2\,{\beta_0}^2\,(\beta_0-\gamma_0)\,\pi}\gamma^{(1)}_{D^{(2)}_{S^2,s}}$$

$$B_2 = \frac{-\beta_1\,\gamma_0\,\gamma^{(1)}_{D^{(2)}_{S^2,s}}+\beta_0\,\gamma_1\,\gamma^{(1)}_{D^{(2)}_{S^2,s}}+\beta_0\left(\beta_1\,\gamma^{(1)}_{D^{(2)}_{S^2,s}}-4\,\beta_0\,\gamma^{(2)}_{D^{(2)}_{S^2,s}}\right)}{2\,{\beta_0}^2\,(2\,\beta_0-\gamma_0)\,\pi}$$

$$
\begin{aligned}
\delta D^{(2)}_{S^2,s}(\nu_p)|_{NLL} &= A_1\alpha_s^2(m)\ln(z^{\beta_0}) + A_2\alpha_s^2(m)(z^{\beta_0}-1) + A_3\alpha_s^2(m)\left(z^{2\beta_0}-1\right) \\
&+A_4\alpha_s^2(m)\left(z^{\frac{-13C_A}{6}+\beta_0}-1\right) + A_5\alpha_s^2(m)\left(z^{-2C_A+\beta_0}-1\right) + A_6\alpha_s^2(m)\left(z^{\frac{-25C_A}{6}+2\beta_0}-1\right) \\
&+A_7\alpha_s^2(m)\left(z^{-4C_A+2\beta_0}-1\right) + A_8\alpha_s^2(m)\left(z^{-3C_A+2\beta_0}-1\right) + A_9\alpha_s^2(m)\left(z^{-2C_A+2\beta_0}-1\right) \\
&+A_{10}\alpha_s^2(m)\left(z^{-C_A+2\beta_0}-1\right) + A_{11}\alpha_s^2(m)\left(z^{-\frac{2(C_A-2n_fT_F)}{3}+\beta_0}-1\right) \\
&+A_{12}\alpha_s^2(m)\left(z^{-\frac{2(4C_A-2n_fT_F)}{3}+2\beta_0}-1\right) + A_{13}\alpha_s^2(m)\left(1-z^{\beta_0}-(2-z^{\beta_0})\ln\left(2-z^{\beta_0}\right)\right) \\
&+A_{14}\alpha_s^2(m)\left(-HypergeometricPFQ(\{1,1,-1+\frac{2\,C_A}{\beta_0}\},\{2,2\},\frac{1}{2})\right. \\
&\qquad -\left(-2+z^{\beta_0}\right)\,HypergeometricPFQ(\{1,1,-1+\frac{2\,C_A}{\beta_0}\},\{2,2\},1-\frac{z^{\beta_0}}{2}) \\
&\qquad \left.+\frac{\beta_0\,\left(4-4^{\frac{C_A}{\beta_0}}\,z^{2\beta_0-2\,C_A}\right)\,\log(2-z^{\beta_0})}{-4\,\beta_0+4\,C_A}\right) \\
&+A_{15}\alpha_s^2(m)\left((1-z^{\beta_0})(5+z^{\beta_0})+2(-4+z^{2\beta_0})\ln\left(2-z^{\beta_0}\right)\right)\,,
\end{aligned}
$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$, $z = \left[\frac{\alpha_s(\nu_p)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}}$ and $w = \left[\frac{\alpha_s(\nu_p^2/m)}{\alpha_s(\nu_p)}\right]^{\frac{1}{\beta_0}}$. The coefficients A_i read

$$\begin{aligned}
A_1 &= \frac{3\,C_A\,{C_f}^2\,\pi}{2\,{\beta_0}^2\,\left(6\,\beta_0 - 13\,C_A\right)\,\left(\beta_0 - 2\,C_A\right)^2\,\left(3\,\beta_0 - 2\,C_A + 4\,n_f\,T_F\right)} \\
&\quad \times \left(-\left(\left(6\,\beta_0 - 13\,C_A\right)\,\left(2\,{C_A}^2\,\left(-9\,C_A + 56\,C_f\right) - 2\,\beta_0\,C_A\,\left(40\,C_f + C_A\,\left(-9 + s(s+1)\right)\right)\right.\right.\right. \\
&\quad \left.\left.\left. +3\,{\beta_0}^2\,\left(4\,C_f + C_A\,\left(-6 + s(s+1)\right)\right)\right)\right) + 4\,n_f\,\left(4\,{C_A}^2\,\left(9\,C_A - 74\,C_f\right)\right. \\
&\quad \left.\left. -6\,{\beta_0}^2\,\left(4\,C_f + C_A\,\left(-6 + s(s+1)\right)\right) + \beta_0\,C_A\,\left(196\,C_f + C_A\,\left(-90 + 13\,s(s+1)\right)\right)\right)\,T_F\right)\,, \\
A_2 &= \frac{3\,C_f\,\pi}{52\,{\beta_0}^2\,\left(\beta_0 - 2\,C_A\right)\,\left(C_A - 2\,n_f\,T_F\right)} \\
&\quad \times \left(13\,C_A\,\left(C_A\,C_f\,\left(-9\,C_A + 56\,C_f\right) + \beta_0\,\left(4\,{C_A}^2 + 5\,C_A\,C_f - 16\,{C_f}^2\right)\right)\right. \\
&\quad \left. -4\,\left(2\,C_A\,C_f\,\left(-9\,C_A + 74\,C_f\right) + \beta_0\,\left(26\,{C_A}^2 + 19\,C_A\,C_f - 32\,{C_f}^2\right)\right)\,n_f\,T_F\right)\,, \\
A_3 &= \frac{{C_f}^2\,\pi}{16\,\beta_0}\,, \\
A_4 &= \frac{288\,\left(6\,\beta_0 - 31\,C_A\right)\,{C_f}^2\,\left(5\,C_A + 8\,C_f\right)\,n_f\,\pi\,T_F}{13\,\left(6\,\beta_0 - 13\,C_A\right)^2\,\left(\beta_0 - 2\,C_A\right)\,\left(9\,C_A + 8\,n_f\,T_F\right)}\,, \\
A_5 &= \frac{{C_f}^2\,\pi}{8\,\left(\beta_0 - 2\,C_A\right)^3} \\
&\quad \times \left(4\,\left(\frac{\left(\beta_0 - 2\,C_A\right)}{\beta_0\,\left(6\,\beta_0 - 13\,C_A\right)}\right.\right. \\
&\qquad \times \left(3\,C_A\,\left(30\,{\beta_0}^2 - 109\,\beta_0\,C_A + 42\,{C_A}^2\right) - 4\,\left(6\,\beta_0 - 37\,C_A\right)\,\left(2\,\beta_0 - 7\,C_A\right)\,C_f\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad -2\,(2\,\beta_0 - 7\,C_A)\,C_A\,s - 2\,(2\,\beta_0 - 7\,C_A)\,C_A\,s^2) \\
&+\frac{27\,(\beta_0 - 5\,C_A)\,{C_A}^2}{C_A + n_f\,T_F} + \frac{216\,(2\,\beta_0 - 7\,C_A)\,{C_A}^2\,((\beta_0 - 3\,C_A)\,C_A - 8\,(\beta_0 - 2\,C_A)\,C_f)}{\beta_0\,(6\,\beta_0 - 13\,C_A)\,(3\,\beta_0 - 2\,C_A + 4\,n_f\,T_F)}\Bigg)\,, \\
A_6 &= \frac{96\,(6\,\beta_0 - 31\,C_A)\,(2\,\beta_0 - 7\,C_A)\,{C_f}^2\,(5\,C_A + 8\,C_f)\,n_f\,\pi\,T_F}{13\,(12\,\beta_0 - 25\,C_A)\,(6\,\beta_0 - 13\,C_A)\,(\beta_0 - 2\,C_A)\,C_A\,(9\,C_A + 8\,n_f\,T_F)}\,, \\
A_7 &= \frac{{C_f}^2\,\pi}{32\,(\beta_0 - 2\,C_A)^3\,(C_A + n_f\,T_F)} \\
&\times\big(-2\,{C_A}^3\,(-371 + 62\,s\,(1+s)) - 4\,{C_A}^2\,n_f\,(-28 + 31\,s\,(1+s))\,T_F \\
&-{\beta_0}^2\,(C_A\,(-52 + 9\,s\,(1+s)) + n_f\,(-16 + 9\,s\,(1+s))\,T_F) \\
&+2\,\beta_0\,C_A\,(C_A\,(-197 + 34\,s\,(1+s)) + 2\,n_f\,(-22 + 17\,s\,(1+s))\,T_F)\big)\,, \\
A_8 &= \frac{C_A\,C_f\,\pi}{-8\,\beta_0 + 12\,C_A}\,, \\
A_9 &= \frac{C_f\,\pi}{104\,\beta_0\,(\beta_0 - 2\,C_A)\,(\beta_0 - C_A)\,C_A\,(C_A - 2\,n_f\,T_F)} \\
&\times\big(13\,C_A\,(4\,\beta_0\,(4\,\beta_0 - 11\,C_A)\,{C_A}^2 + C_A\,(-10\,{\beta_0}^2 - 13\,\beta_0\,C_A + 63\,{C_A}^2)\,C_f \\
&\qquad -8\,(2\,\beta_0 - 7\,C_A)^2\,{C_f}^2\big) + 4\,\big(-26\,\beta_0\,(4\,\beta_0 - 11\,C_A)\,{C_A}^2 \\
&\qquad\quad + C_A\,(92\,{\beta_0}^2 - 91\,\beta_0\,C_A - 126\,{C_A}^2)\,C_f \\
&\qquad\quad + 4\,(8\,\beta_0 - 37\,C_A)\,(2\,\beta_0 - 7\,C_A)\,{C_f}^2\big)\,n_f\,T_F\big)\,, \\
A_{10} &= -\frac{(C_A - 3\,C_f)\,C_f\,\pi}{2\,\beta_0 - C_A}\,,
\end{aligned}
$$

$$A_{11} = \frac{81\, {C_A}^2\, {C_f}^2\, \pi\, \left(-3\, \beta_0 + 11\, C_A - 4\, n_f\, T_F\right)\, \left(C_A\, \left(7\, C_A - 32\, C_f\right) + 4\, \left(C_A - 8\, C_f\right)\, n_f\, T_F\right)}{8\, \left(\beta_0 - 2\, C_A\right)\, \left(C_A - 2\, n_f\, T_F\right)\, \left(C_A + n_f\, T_F\right)\, \left(3\, \beta_0 - 2\, C_A + 4\, n_f\, T_F\right)^2\, \left(9\, C_A + 8\, n_f\, T_F\right)},$$

$$A_{12} = \frac{-27\, \left(2\, \beta_0 - 7\, C_A\right)\, C_A\, {C_f}^2\, \pi\, \left(3\, \beta_0 - 11\, C_A + 4\, n_f\, T_F\right)}{16\, \left(\beta_0 - 2\, C_A\right)\, \left(C_A - 2\, n_f\, T_F\right)\, \left(C_A + n_f\, T_F\right)\, \left(3\, \beta_0 - 4\, C_A + 2\, n_f\, T_F\right)}$$

$$\times \frac{\left(C_A\, \left(7\, C_A - 32\, C_f\right) + 4\, \left(C_A - 8\, C_f\right)\, n_f\, T_F\right)}{\left(3\, \beta_0 - 2\, C_A + 4\, n_f\, T_F\right)\, \left(9\, C_A + 8\, n_f\, T_F\right)}$$

$$A_{13} = \frac{8\, C_f\, C_A\, (C_A^2 + 3C_A C_f + 2C_f^2)\pi}{\beta_0^2\, (\beta_0 - 2\, C_A)}$$

$$A_{14} = \frac{8\, C_f\, (C_f + C_A)\, (C_A + 2C_f)(2\beta_0 - 7C_A) 2^{1 - \frac{2C_A}{\beta_0}}\, \pi}{3\, \beta_0^2\, (\beta_0 - 2\, C_A)},$$

$$A_{15} = \frac{C_A^3 \pi}{4\beta_0^2}\, .$$

Decay Ratio at NNLL

Penin, Smirnov, Steinhauser, Pineda

$$\frac{\Gamma(V_Q(nS) \to e^+e^-)}{\Gamma(P_Q(nS) \to \gamma\gamma)} \sim 1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \cdots$$

$$+\alpha + \alpha^2 \ln \alpha + \alpha^3 \ln^2 \alpha + \cdots$$

$$+\alpha^2 + \alpha^3 \ln \alpha + \alpha^4 \ln^2 \alpha + \cdots$$

$$\frac{\Gamma(T(1S) \to e^+e^-)}{\Gamma(\eta_t(1S) \to \gamma\gamma)} = \frac{1}{3Q_t^2} (1 - 0.13198 - 0.0179492) \; .$$

$$\frac{\Gamma(\Upsilon(1S) \to e^+e^-)}{\Gamma(\eta_b(1S) \to \gamma\gamma)} = \frac{1}{3Q_b^2} (1 - 0.302 - 0.111) \; .$$

$$\frac{\Gamma(J/\Psi(1S) \to e^+e^-)}{\Gamma(\eta_c(1S) \to \gamma\gamma)} = \frac{1}{3Q_c^2} (1 - 0.51313 - 0.325764) \; .$$

$$\Gamma(\eta_b(1S) \to \gamma\gamma) = 0.659 \pm 0.089(\text{th.})^{+0.019}_{-0.018}(\delta\alpha_{\text{s}}) \pm 0.015(\text{exp.}) \text{ KeV} \, ,$$

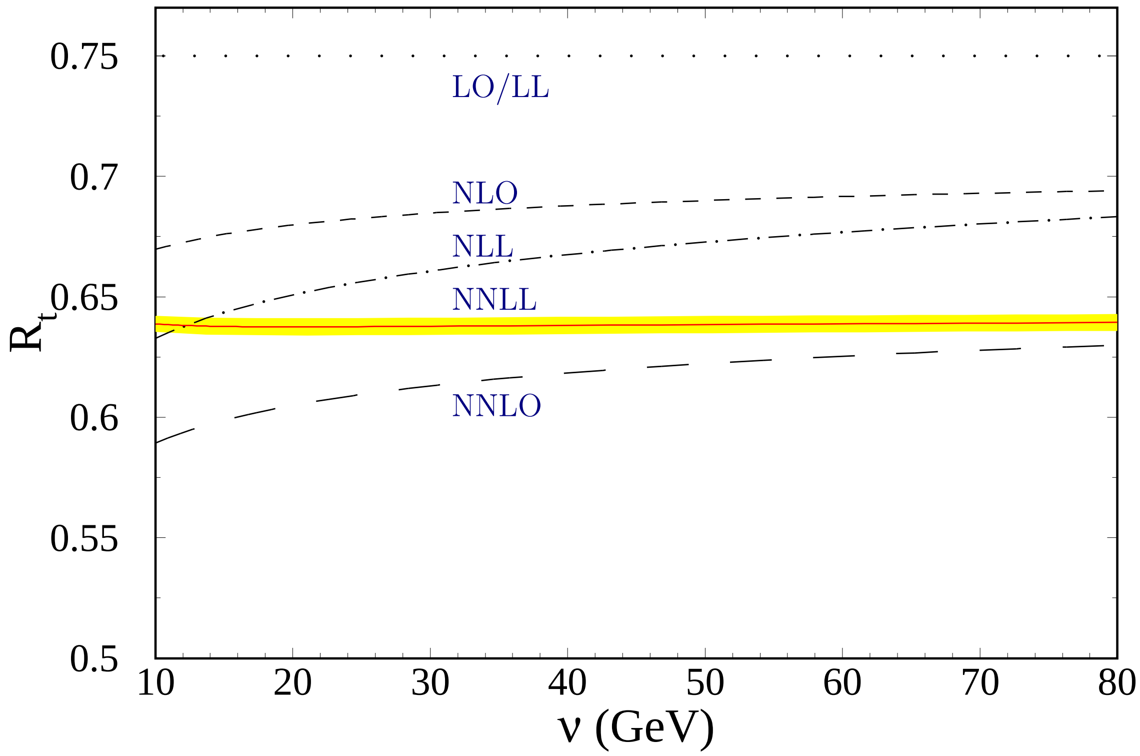


Figure 4: The spin ratio as the function of the renormalization scale ν in LO (dotted line), NLO (short-dashed line), NNLO (long-dashed line), LL (dotted line), NLL (dot-dashed line), and NNLL (bold solid line) approximation for the (would be) toponium ground state with $\nu_h = m_t$. For the NNLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$

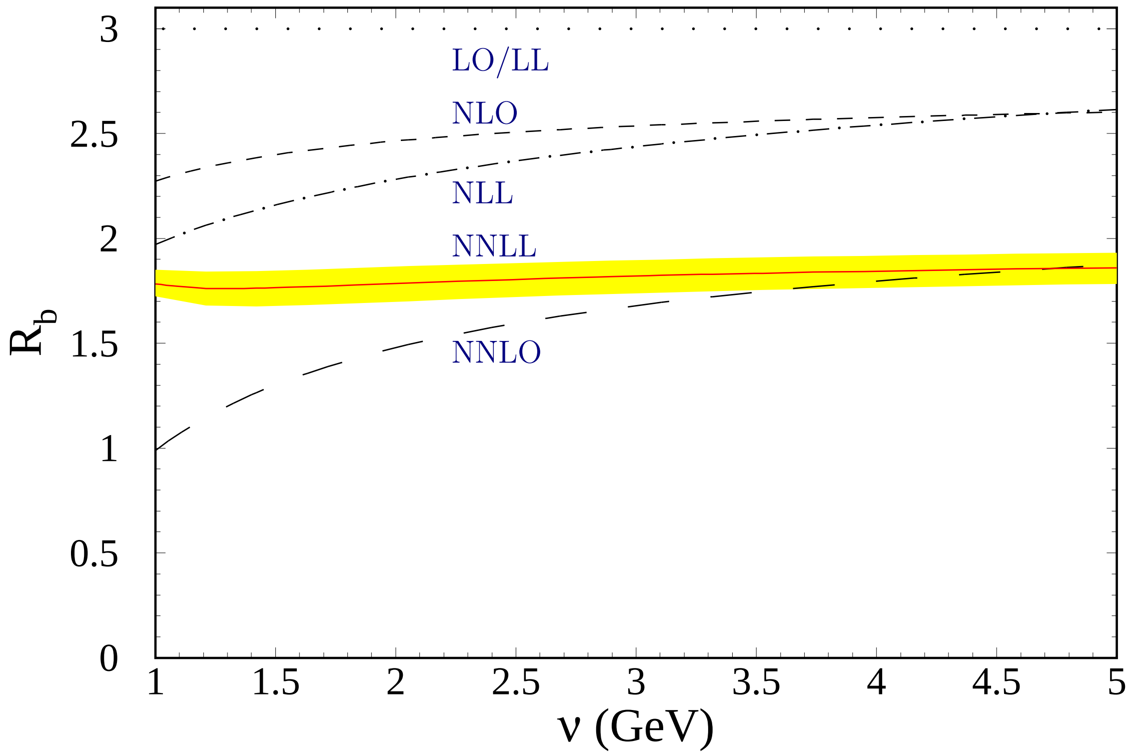


Figure 5: The spin ratio as the function of the renormalization scale ν in LO (dotted line), NLO (short-dashed line), NNLO (long-dashed line), LL (dotted line), NLL (dot-dashed line), and NNLL (bold solid line) approximation for the bottomonium ground state with $\nu_h = m_b$. For the NNLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$

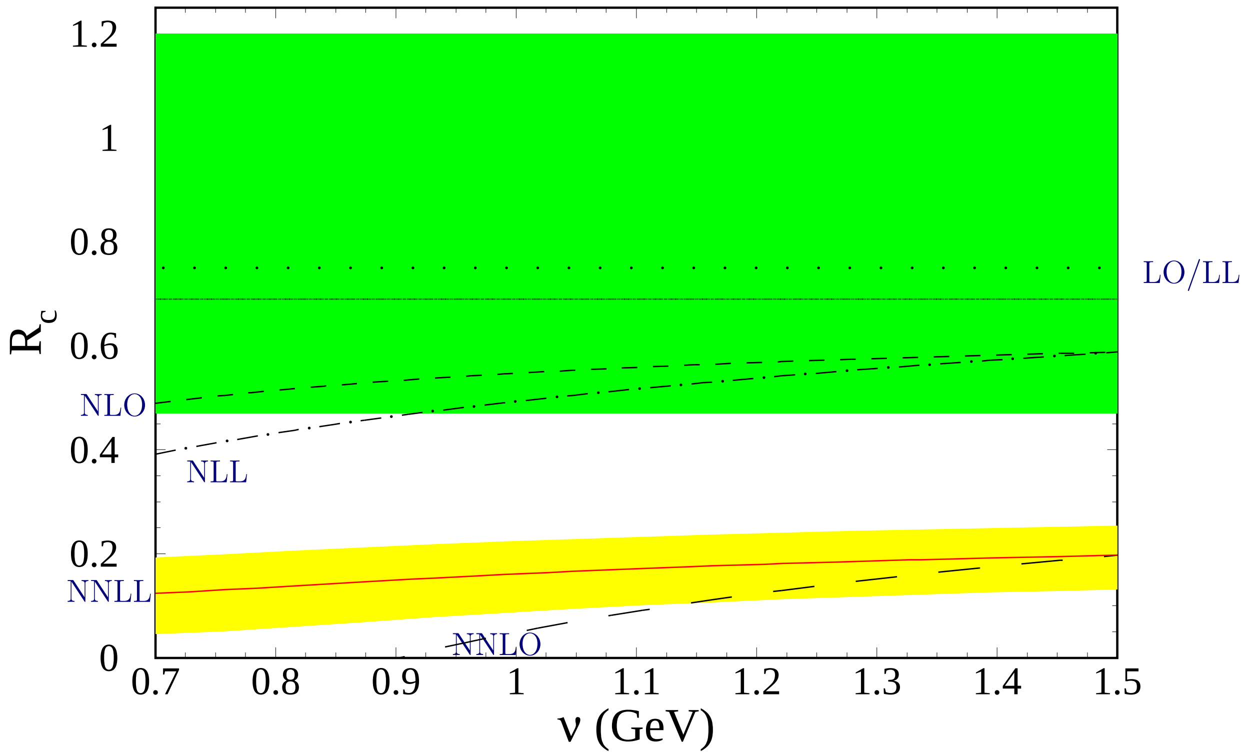


Figure 6: The spin ratio as the function of the renormalization scale ν in LO (dotted line), NLO (short-dashed line), NNLO (long-dashed line), LL (dotted line), NLL (dot-dashed line), and NNLL (bold solid line) approximation for the charmonium ground state with $\nu_h = m_c$. For the NNLL result the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$. The horizontal band represents the experimental error of the ratio.

Non-relativistic Sum rules: bottomonium

Pineda-Signer

$$M_n \equiv \frac{12\pi^2 e_b^2}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)|_{q^2=0} = \int_0^\infty \frac{ds}{s^{n+1}} R_{b\bar{b}}(s),$$

$$M_n = 48\pi e_b^2 N_c \int_{-\infty}^{\infty} \frac{dE}{(E+2m_b)^{2n+3}}\left(c_1^2 - c_1 c_2 \frac{E}{3m_b}\right) \mathrm{Im}\, G(0,0,E)$$

n	$m_{b,\mathrm{PS}}(2\ \mathrm{GeV})$	Δ_{th}	Δ_{exp}	Δ_α	Δ_{tot}	$\overline{m}_b$
6	4460	40	50	35	70	4135 ± 65
8	4505	45	25	30	60	4170 ± 55
10	4515	45	15	25	55	4185 ± 50
12	4520	45	10	20	50	4185 ± 45
14	4520	40	10	15	45	4185 ± 40
n	$m_{b,\mathrm{RS}}(2\ \mathrm{GeV})$	Δ_{th}	Δ_{exp}	Δ_α	Δ_{tot}	$\overline{m}_b$
6	4315	55	50	25	80	4140 ± 70
8	4360	65	30	20	75	4180 ± 65
10	4370	65	20	10	70	4190 ± 60
12	4370	65	15	5	65	4190 ± 60
14	4370	65	10	5	65	4185 ± 55

Table 1: *Extraction of $m_{b,\mathrm{PS/RS}}$(2 GeV) with errors for various n. All values are given in MeV and rounded to 5 MeV. The total error has been obtained by adding the partial errors in quadrature. The corresponding value for the $\overline{\mathrm{MS}}$ mass with its error is given in the last column.*

$$m_{b,\mathrm{PS}}(2\mathrm{GeV}) = 4.52 \pm 0.06\ \mathrm{GeV}, m_{b,\mathrm{RS}}(2\mathrm{GeV}) = 4.37 \pm 0.07\ \mathrm{GeV}.$$

$$\overline{m}_b(\overline{m}_b) = 4.19 \pm 0.06\,\mathrm{GeV}.$$

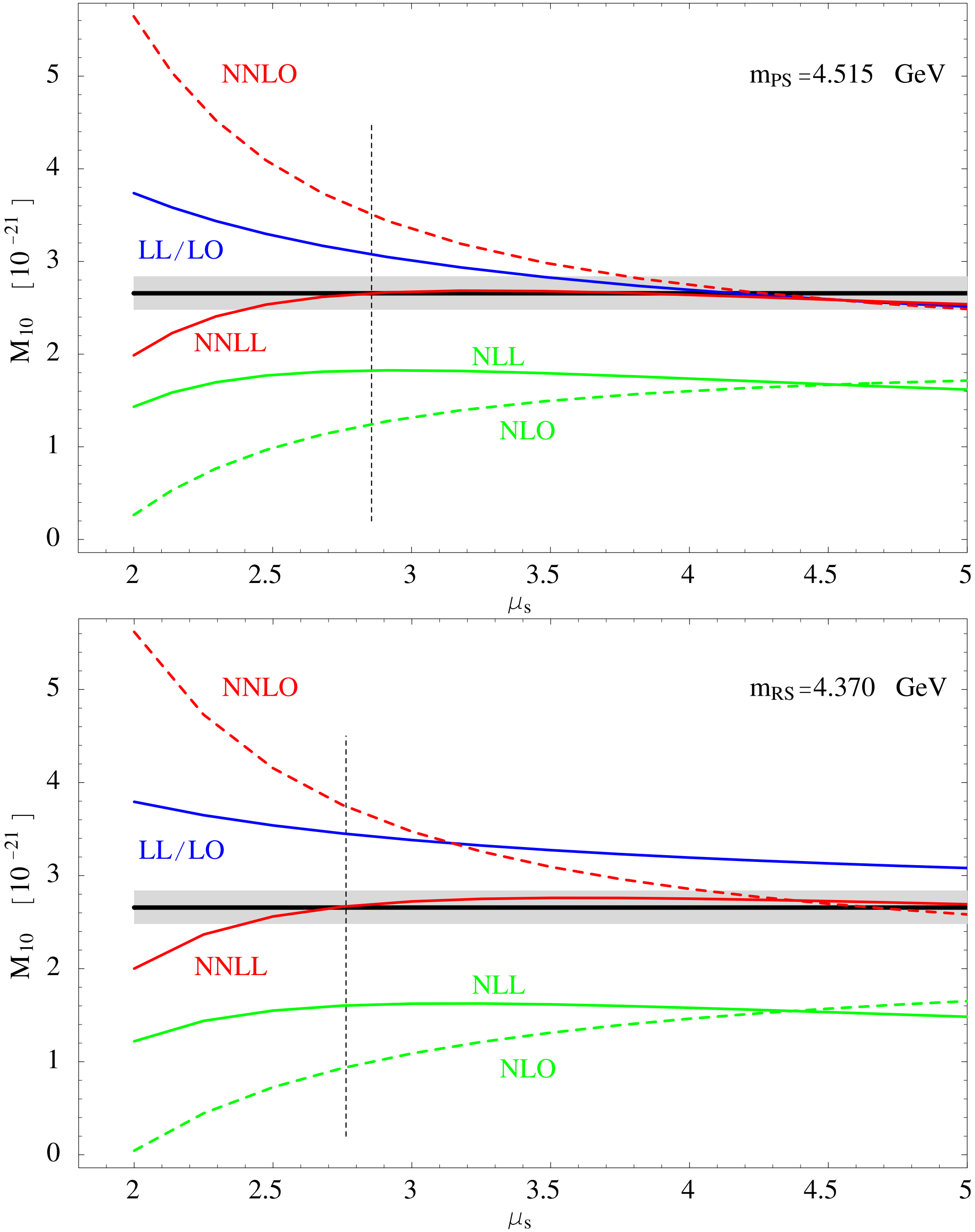


Figure 7: *The moment M_{10} as a function of μ_s at LO/LL, NLO, NLL, NNLO and NNLL for $m_{b\mathrm{PS}}(2\ \mathrm{GeV}) = 4.515$ GeV in the PS scheme (upper figure), and for $m_{b\mathrm{RS}}(2\ \mathrm{GeV}) = 4.370$ GeV in the RS scheme (lower figure). The experimental moment with its error is also shown (grey band).*

The top mass

Next Linear Collider. $\delta m_t(\text{exp.}) \lesssim 30$ MeV; decay width 2%: Martinez-Miquel

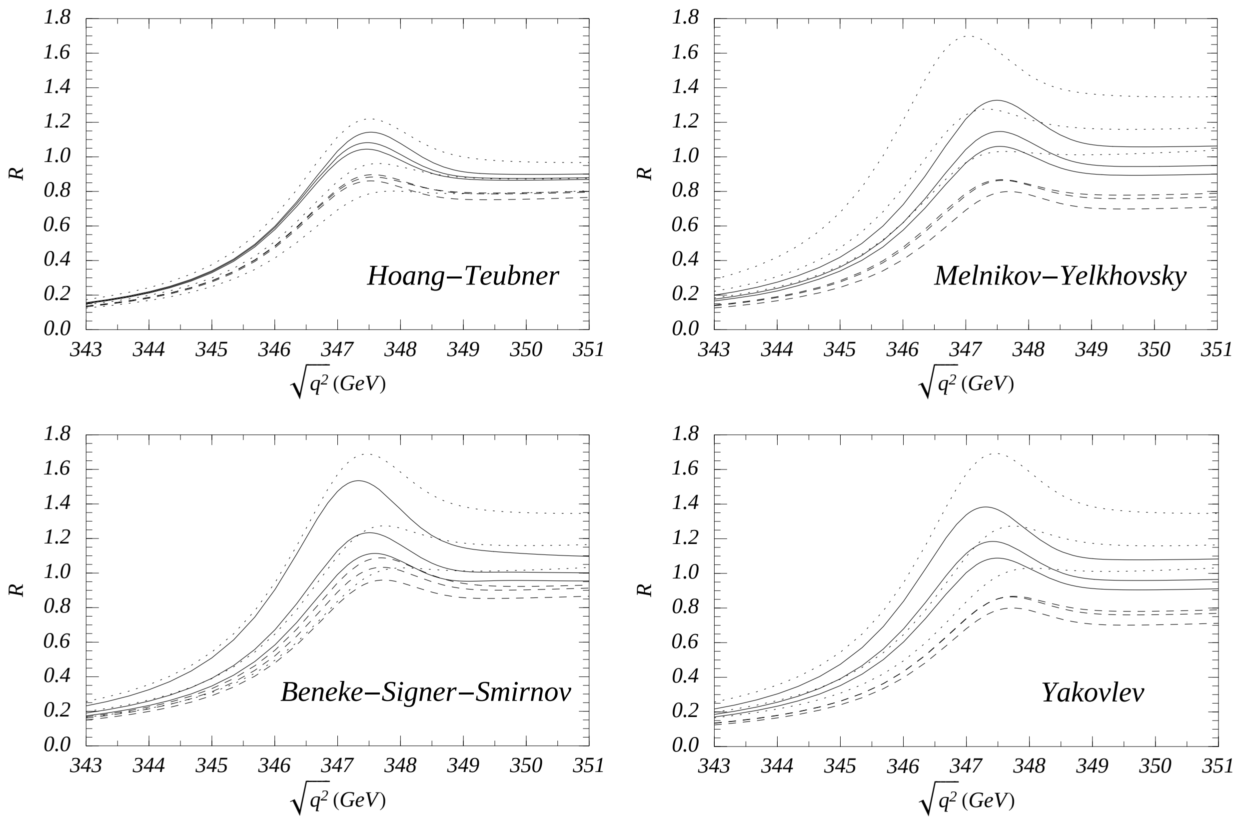
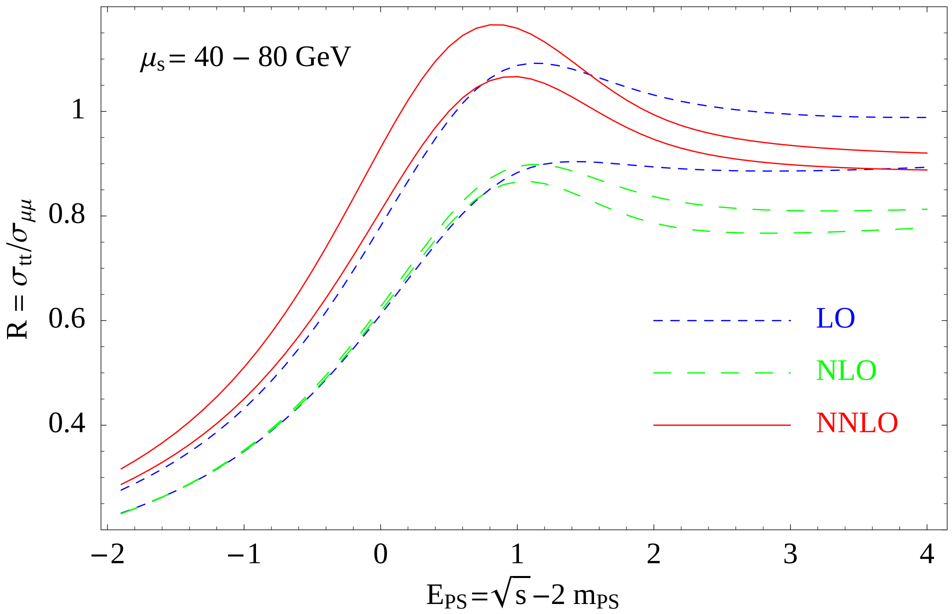


Figure 8:

Total normalized photon induced $t\bar{t}$ cross section at the International Linear Collider versus the center of mass energy at LO (dotted line), NLO (dashed line) and NNLO (solid line). Hoang-Teubner used the 1S scheme with $m_t^{1S} = 173.68$ GeV, Melnikov–Yelkhovsky the kinetic mass $m_{t,15\,\text{GeV}}^{\text{kin}} = 173.10$ GeV, and Beneke–Signer–Smirnov and Yakovlev the PS massa $m_{t,20\,\text{GeV}}^{\text{PS}} = 173.30$ GeV. Plot from hep-ph/0001286.



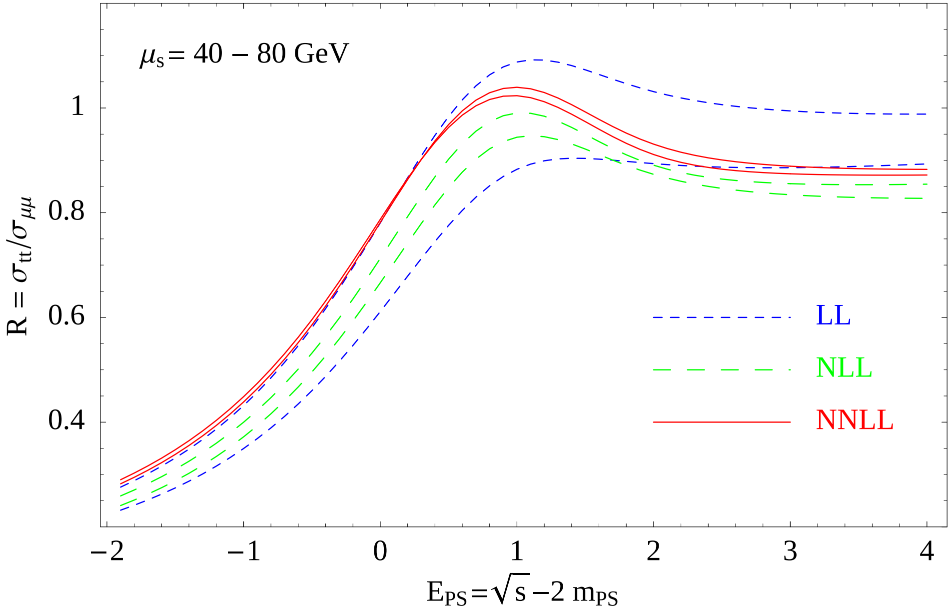


Figure 9: Threshold scan for $t\bar{t}$ using $m_{t,PS} = 175$ GeV. The upper figure shows the fixed order results, LO, NLO and NNLO, whereas the figure belows the RGI results LL, NLL and NNLL are displayed. The soft scale is varied from μ_s=40 GeV to μ_s=80 GeV

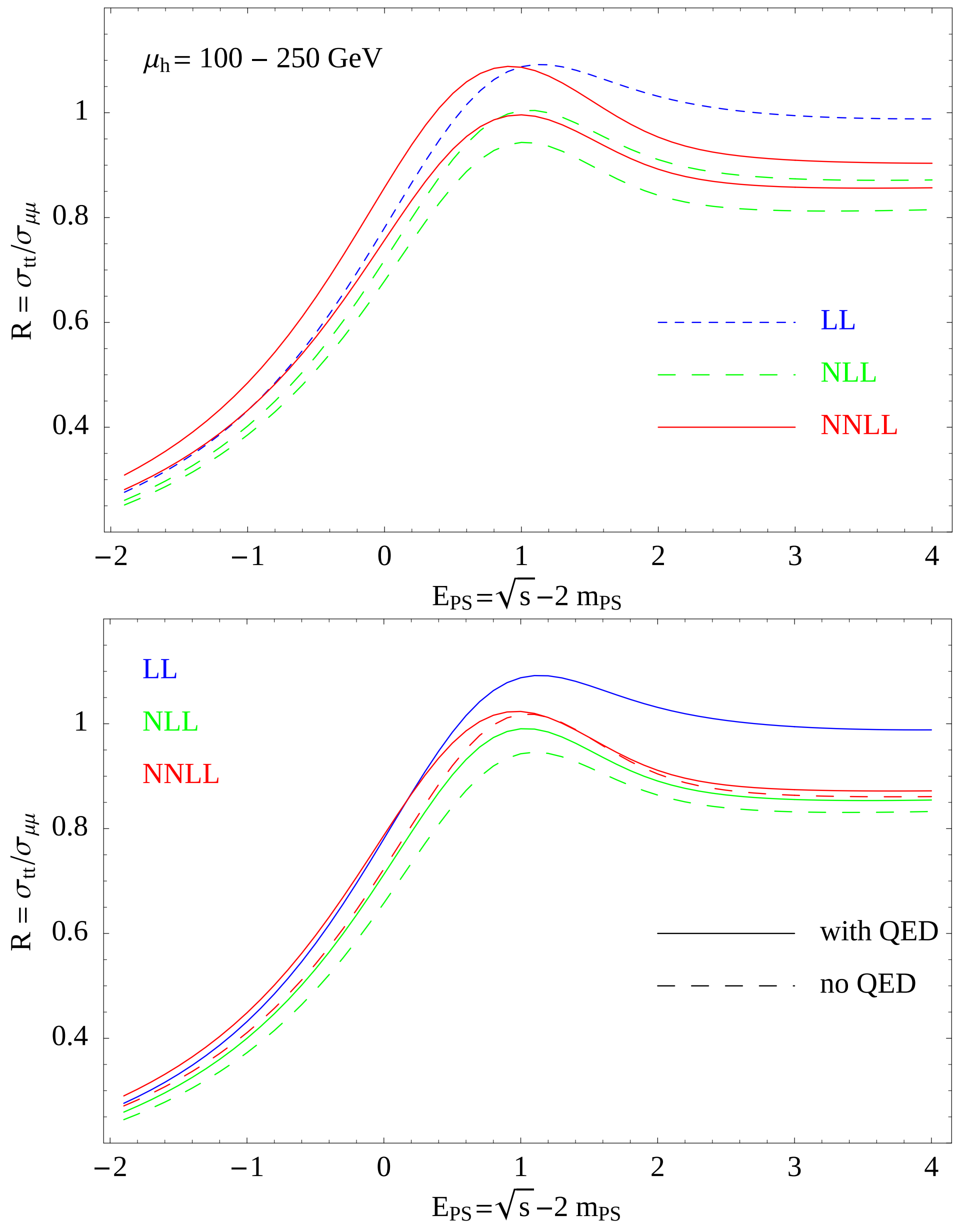


Figure 10: Upper panel: Variation of the hard scale μ_h in the threshold scan for $m_{t,PS} = 175$ GeV. The hard scale is varied from 100 GeV $\leq \mu_h \leq$ 250 GeV. Lower panel: the effect of including QED corrections at NLL and NNLL. There are no QED corrections at LL.

Conclusions

Potentials at LL and $D^{(2)}_{S^2,s}$ at NLL

Heavy quarkonium at NNLL: $m\alpha^{4+n}\ln^n\alpha$

Hyperfine splitting at NLL: η_b, η_c, B_c

B_s at NLL

$\Gamma(V_Q(nS)\to e^+e^-)$, $\Gamma(P_Q(nS)\to\gamma\gamma)$ at NLL

$\Gamma(V_Q(nS)\to e^+e^-)$ and $\Gamma(P_Q(nS)\to\gamma\gamma)$ ratio at NNLL

Applications: b-$\bar{b}$ (c-$\bar{c}$?) sum rules, t-$\bar{t}$ production, inclusive production/annihilation.

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Prospects

Potentials at NLL (some already known)

B_s at NNLL. Applications: b-$\bar{b}$ (c-$\bar{c}$?) sum rules, t-$\bar{t}$ production, inclusive production/annihilation.

$\Gamma(V_Q(nS)\to e^+e^-)$, $\Gamma(P_Q(nS)\to\gamma\gamma)$ at NNLL

Very precise determinations of m_b, m_t and α_s!! (also top-Higgs coupling).

Heavy quarkonium at NNNLL(?): $m\alpha^{5+n}\ln^n\alpha$ ($O(m\alpha^5)$ almost known: Kniehl, Penin, Smirnov, Steinhauser; Penin, Steinhauser)

Caveat of non-perturbative/ultrasoft effects (b-$\bar{b}$ physics)

Caveat on renormalon effects.